Institut für Angewandte Mathematik
Markov Processes

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Exercise sheet 5

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Exercise 1 (Ornstein-Uhlenbeck's Process) (4 Points)

The semigroup of the Ornstein-Uhlenbeck Process is given by:

$$(P_t f)(x) = \int f(e^{-t}x + \sqrt{1-e^{-2t}}y)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy,$$

for $f \in C_0(\mathbb{R})$. Show that the infinitesimal generator of the process is given by

$$Gf(x) = f''(x) - xf'(x),$$

and find its domain.

Exercise 2 (The Translation Semigroup) (8 Points)

For the Banach space $C_0(\mathbb{R})$ consider the semigroup $\{P_t\}_{t\in\mathbb{R}_+}$ defined by $(P_t f)(x) = f(x+t)$ for each $f \in C_0(\mathbb{R}), \quad x \in \mathbb{R}, \quad t \geq 0$. Prove that:

(a) P_t is a strongly continuous contraction semigroup;

(b) Find the infinitesimal generator G and its domain;

(c) Prove that G is a closed linear operator.

Exercise 3 (8 Points)

Let $\{X_t\}_{t\in\mathbb{R}_+}$ be a one dimensional Lévy process with characteristic triplet $(0,0,\nu)$, where $\nu(-\infty,0) = 0$ and $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$.
Prove that the infinitesimal generator G of X takes the following form

$$(Gf)(x) = \int_0^\infty (f(x+y) - f(x))\nu(dx),$$

for all $f \in C_0^1(\mathbb{R})$.

Sum: 20 Points
Note: The Banach spaces given in the exercise are equipped with the sup-norm, except if otherwise stated.