Institut für Angewandte Mathematik
Markov Processes

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Exercise sheet 4

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Exercise 1 **(8 Points)**

Let $(X_t)_{t\geq 0}$ be a Lévy process with $X_0 = 0$, and let μ_t denote the law of X_t.

a) Prove that $\mu_{t+s} = \mu_t * \mu_s$ for $t, s \geq 0$.

For bounded measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ and $t \geq 0$ we define

$$P_t f(x) := \int_{\mathbb{R}^d} f(x+y)\, d\mu_t(y) \ .$$

b) Show, for fixed $t > 0$, that $P_t f$ is continuous for all bounded measurable functions f if and only if μ_t is absolutely continuous with respect to the Lebesgue measure.

Exercise 2 **(4 Points)**

A linear operator $G : \mathcal{D}(G) \subseteq \mathcal{B} \to \mathcal{B}$ on a Banach space $\mathcal{B}$ is called *closable* if the closure of its graph $\{(x, Gx) \ : \ x \in \mathcal{D}(G)\}$ is again the graph of a linear operator.

a) Show that G is closable if and only if

$$\big[(f_n)_n \subseteq \mathcal{D}(G),\ f_n \to 0,\ Gf_n \to g\big] \ \Rightarrow\ [g = 0].$$

b) Give an example of a linear operator G that is not closable.

Exercise 3 **(Hille-Yosida)** **(8 Points)**

Let $B_0 = C_0(\mathbb{R})$ be a Banach space equipped with the sup-norm and $G = \frac{1}{2}\frac{d^2}{dx^2}$ be a linear operator with $\mathcal{D}(G) = C_0^2(\mathbb{R})$.

(a) Verify that for

$$(R_\lambda f)(x) = \frac{1}{\sqrt{2\lambda}} \int_{-\infty}^{\infty} f(y) e^{-\sqrt{2\lambda}|x-y|} dy,$$

the linear operator $(\lambda I - G)$ is the inverse of R_λ.

(b) Use the *Hille-Yosida* Theorem to prove that there exists a unique strongly continuous contraction semi-group, P_t , $t \in \mathbb{R}$, such that

$$\int_0^\infty e^{-\lambda t} P_t f dt = R_\lambda f,$$

for all $\lambda > 0$ and for all $f \in C_0(\mathbb{R})$.

(c) Prove that

$$(P_t f)(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} f(y) e^{-\frac{(x-y)^2}{2t}} dy,$$

for all $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$.

Sum: 20 Points