Institut für Angewandte Mathematik
Markov Processes

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Exercise sheet 3

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Exercise 1 **(4 Points)**

Let $(X_t)_{t\geq 0}$ be a Markov process with transition function $(P_t)_{t\geq 0}$ and f a bounded measurable function. Prove that $(P_{t-s}f(X_s), s \in [0,t])$ is a martingale for any $t > 0$.

Exercise 2 (Probabilistic interpretation of the resolvent) **(6 Points)**

Let $(P_t)_{t\geq 0}$ be the transition kernel of a continuous time Markov process. Furthermore, let S and T be independent exponentially distributed random variables, taking values in $[0,\infty)$ with rate λ and μ respectively. Prove that for $f \in B(\mathbb{R})$ and $\lambda \neq \mu$

(a) $E[(P_S f)(x)] = \lambda(R_\lambda f)(x)$;

(b) $P(S+T \in du) = \lambda\mu \frac{e^{-\lambda u} - e^{-\mu u}}{\mu - \lambda} du$;

(c) $E[P_{S+T} f] = \lambda\mu R_\lambda R_\mu f$;

Using the previous results prove the *resolvent identity*.

Exercise 3 **(6 Points)**

Prove that the transition kernel of a one dimensional Brownian motion is given by

$$P_t(x,A) = \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{(y-x)^2}{2t}\right) dy, \text{ for } t > 0,$$

where $x \in \mathbb{R}$ and$A \in \mathcal{B}(\mathbb{R})$. Assume that initial distribution is $P_0(x,\cdot) = \delta_x(\cdot)$, where δ_x denotes the Dirac measure for all $x \in \mathbb{R}$. Verify that the transition kernel $(P_t)_{t\geq 0}$ defines an honest sub-Markov semi-group.

Exercise 4 **(6 Points)**

Let $(X_t)_{t\geq 0}$ be a stochastic process, let $(P_t)_{t\geq 0}$ be a transition function and let ν be a probability measure on E. Prove that the following assertions are equivalent:

(a) $(X_t)_{t\geq 0}$ is a Markov process with transition function $(P_t)_{t\geq 0}$ and initial distribution ν with respect to $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \sigma(X_s, s \leq t)$;

(b) For any $0 = t_0 < t_1 < \ldots < t_k$ and all bounded measurable functions $f_0, \ldots, f_k : E \to \mathbb{R}$,

$$E\left[\prod_{i=0}^{k} f_i(X_{t_i})\right] = \int_E \nu(dx_0) f_0(x_0) \int_E P_{t_1}(x_0, dx_1) f_1(x_1) \ldots \int_E P_{t_k - t_{k-1}}(x_{k-1}, dx_k) f_k(x_k).$$

Exercise 5 **(8 Points)**

Let $\{N_t\}_{t\in\mathbb{R}_+}$ be a Poisson process with intensity $\lambda > 0$, and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d random variables with distribution function $F(x)$, independent of N. We define $S_n = X_1 + \cdots + X_n$ and the compound Poisson process Y by

$$Y_t = \sum_{n=1}^{N_t} X_n.$$

Furthermore, let $B(\mathbb{R})$ be a Banach space with respect to the norm

$$\|f\| = \sup\{|f(x)|, x \in \mathbb{R}\},$$

for each $f \in B(\mathbb{R})$.

Prove that:

(a) For all $f \in B(\mathbb{R})$, the operator L defined by

$$Lf(x) = E[f(x + X_1)] = \int_{\mathbb{R}} f(x + y)F(dy),$$

is a bounded operator on $B(\mathbb{R})$,

(b) For the the operator $G = \lambda(L - I)$ on $B(\mathbb{R})$, we have $(P_t f)(x) = \left(e^{Gt} f\right)(x)$, where I is the identity operator.

(c) The linear operator $(P_t)_{t \geq 0}$, defined by

$$(P_t f)(x) = E[f(Y_t)|Y_0 = x] = E[f(x + Y_t)],$$

for $f \in B(\mathbb{R})$, is a strongly continuous contraction semigroup.

Hint: For part (b) use the fact that $L^n f(x) = E[f(x + X_1 + \cdots + X_n)] = E[f(x + S_n)]$.

Sum: 30 Points