Institut für Angewandte Mathematik
Markov Processes

Prof. Dr. A. Bovier / Dr. E. Petrou

universitätbonn

Exercise sheet 2

Ausgabe: 17.04.2012
Abgabe: 24.04.2012

Exercise 1 **(4 Points)**

Consider the continuous time Markov chain with state space $S = \{0, 1\}$ and generator G

$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix},$$

where $\alpha, \beta > 0$. Find the corresponding transition function.

Exercise 2 **(4 Points)**

Let $(E, \mathcal{E})$ be a Polish space endowed with its Borel σ-algebra. Let $\pi = \pi^1$ be a transition kernel on $(E, \mathcal{E})$ and define inductively

$$\pi^n(x, B) = \int_E \pi(x, dy)\pi^{n-1}(y, B)$$

for $x \in E$ and $B \in \mathcal{E}$. Prove that for any $t \geq 0$

$$P_t(x, B) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \pi^n(x, B),$$

defines a transition function and describe the corresponding Markov process.

Exercise 3 **(12 Points)**

Let X be a discrete time Markov chain on $\mathbb{Z}$ associated to the transition matrix Q given by

$$P(x, x+1) = p, P(x, x) = r, P(x, x-1) = q,$$

where $p + r + q = 1, p > 0, q > 0, r \geq 0$. We set $\rho = \frac{q}{p}$.
Fix $a, b \in \mathbb{Z}$, such that $a < b - 1$ and denote by $\tau = \inf\{n \geq 0 : X_n \notin (a, b)\}$ the hitting time in $\{a+1, a+2, \ldots, b-1\}^c$. Let $\Phi_{a,b}$ be the set of applications from $\{a, a+1, \ldots, b\}$ into $\mathbb{R}$.

a). In the following we prove that, if $g \in \Phi_{a,b}$, the system

$$\begin{cases} (I - P)u(x) = g(x) & a < x < b \\ u(a) = \alpha, & u(b) = \beta \end{cases} \tag{1}$$

has a unique solution for every $\alpha, \beta \in \mathbb{R}$.

i). Prove that there exists a unique $\phi \in \Phi_{a,b}$ such that $\phi(a) = 0$, $\phi(a+1) = 1$, $(I - P)\phi(x) = 0$ for every $a < x < b$, and that such a ϕ is strictly increasing.

ii). Prove that there exists a unique $\psi \in \Phi_{a,b}$ such that $\psi(a) = \psi(a+1) = 0$ and $(I - P)\psi(x) = g(x)$ for every $a < x < b$.

iii). Show that, if $u \in \Phi_{a,b}$ is such that $u(a) = u(b) = 0$ and $(I - P)u(x) = 0$ for every $a < x < b$, then $u \equiv 0$.

iv). Prove that there exists a unique function $u \in \Phi_{a,b}$ satisfying (1).

b). Let $u_1 \in \Phi_{a,b}$ be the solution of

$$\begin{cases} (I-P)u_1(x) = 1 & a < x < b \\ u_1(a) = u_1(b) = 0 \end{cases} \tag{2}$$

Show that, for every $x \in (a,b)$,

$$E_x[u_1(X_{n\wedge\tau})] - u_1(x) = -E_x[n \wedge \tau].$$

Deduce that $E_x[\tau] < \infty$ and that $E_x[\tau] = u_1(x)$, $x \in (a,b)$.

c). Let u be the solution of (1). Show that, for every $x \in (a,b)$,

$$u(x) = \alpha P_x(X_\tau = a) + \beta P_x(X_\tau = b) + E_x\left[\sum_{k=0}^{\tau-1} g(X_k)\right].$$

d). (Computation of Moments)
Let us assume for the time being that for every $s > 0$ and every x, $E_x[\tau^s] < \infty$. Let us set $u_s(x) = E_x(\tau^s)$.

i). Show that, for every $x \in (a,b)$, $\tau = 1 + \tau \circ \theta_1$ P_x-a.s.

ii). Prove that u_2 is a solution of (1) for some α, β and g to be determined as functions of u_1. Respectively, prove that u_3 is a solution of (1) for some α, β and g to be determined as functions of u_1 and u_2.

Sum: 20 Points