Institut für Angewandte Mathematik
Markov Processes

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Exercise sheet 1

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Exercise 1 **(- Points)**

Let $(S, \mathcal{B})$ be a Polish space and let $\mathcal{P}$ be a transition kernel(see Definition 1.1) and P_0 a probability measure on $(S, \mathcal{B})$. Prove that there exists a unique a discrete time Markov process with transition kernel $\mathcal{P}$ and $\mathbb{P}(X_0 \in A) = \mathbb{P}_0(A)$, for all $A \in \mathcal{B}$.
Hint: Use Kolmogorov's extension Theorem.

Exercise 2 **(3 Points)**

Show by example that a function $f(X_0), f(X_1), \ldots$ of a Markov chain need not be a Markov chain.

Exercise 3 **(5 Points)**

Consider a simple random walk on $\{-N, -N+1, \ldots, N\}$. Assume we want to condition this process on hitting $+N$ before $-N$. Then let

$$h(x) = \mathbb{P}_x[\tau_N = \tau_{\{N\} \cup \{-N\}}] = \mathbb{P}_x[\tau_N < \tau_{-N}].$$

Compute $h(x)$ and use this to compute the transition rates of the h-transformed walk.

Exercise 4 **(12 Points)**

Let X be Markov Process on a countable set S, with transition matrix P.

(a). Let f be a positive function on S. For $B \subset S$, let I_B be a matrix such that

$$I_B(x,x) = 1 \text{ if } x \in B, \quad I_B(x,x) = 0 \text{ if } x \in B^c, \quad I_B(x,y) = 0 \text{ if } x \neq y$$

i). Show that

$$I_B f(x) = \mathbf{1}_B(x) f(x) \text{ and } (I_B P) f(x) = E_x[\mathbf{1}_B(X_0) f(X_1)]$$

ii). Show that, for every $n \geq 1$

$$(I_B P)^n f(x) = E_x[\mathbf{1}_B(X_0)\mathbf{1}_B(X_1)\ldots\mathbf{1}_B(X_{n-1}) f(X_n)]$$

b). Let $A \subset S$ and $\tau = \inf\{n \geq 0 : X_n \in A\}$. We define

$$P_A f(x) = E_x[\mathbf{1}_{\tau<\infty} f(X_\tau)], \quad U_A f(x) = \mathbf{1}_{A^c} E_x\left[\sum_{n=0}^{\tau-1} f(X_n)\right],$$

for non-negative functions f on S.

i). Show that, for every $n \geq 0$,

$$E_x[\mathbf{1}_{\tau=n} f(X_n)] = (I_{A^c} P)^n I_A f(x)$$

and deduce that

$$P_A = \sum_{n\geq 0} (I_{A^c} P)^n I_A.$$

ii). Show that, for every $n \geq 0$,

$$E_x[\mathbf{1}_{\tau>n} f(X_n)] = (I_{A^c}P)^n I_{A^c} f(x)$$

and deduce that

$$U_A = \sum_{n\geq 0} (I_{A^c}P)^n I_{A^c} = \sum_{n\geq 0} I_{A^c}(I_{A^c}P)^n.$$

c). Show that

$$P_A = I_A + I_{A^c}PP_A, \quad U_A = I_{A^c} + I_{A^c}PU_A$$

d). Let g and h be positive functions on S. We set

$$u = P_A g + U_A h$$

i). Show that $u = I_A g + I_{A^c}(h + Pu)$ and derive that u satisfies

$$u(x) = \begin{cases} g(x) & \text{on } A \\ h(x) + Pu(x) & \text{on } A^c \end{cases} \tag{1}$$

ii). Let v be another solution of (1). Show that, for every $n \geq 0$,

$$v \geq \sum_{k=0}^{n} (I_{A^c}P)^k (I_A g + I_{A^c} h)$$

and deduce that $v \geq u$.

iii). Show that $u(x) = E_x[\tau]$ is the smallest positive solution of

$$u(x) = \begin{cases} 0 & \text{on } A \\ 1 + Pu(x) & \text{on } A^c \end{cases}$$

Sum: 20 Points