Stochastische Prozesse   SoSem 2007   Blatt 7

1. (Trefferzeiten für den zweidimensionalen random walk). Z_n sei der random walk auf dem zweidimensionalen Gitter, der in z_0 startet und mit gleicher Wahrscheinlichkeit einen Schritt in eine der vier Richtungen macht.

a) Zeige, dass $|Z_n|^2 - n$ ein Martingal ist.

b) Für $R > 0$ sei
$$T \;:=\; \inf\{n \geq 0 \,|\, |Z_n|^2 \geq R^2\}$$
die Austrittszeit aus dem Kreis um 0 mit Radius R. Zeige:
$$R^2 - |z_0|^2 \;\leq\; E[T] \;\leq\; (R+1)^2 - |z_0|^2 \,.$$

2. (Labouchere system). Here is a gambling system for playing a fair game. Choose a sequence $x_1, x_2, \ldots, x_n$ of positive numbers. Wager the sum of the first and the last numbers on an evens bet. If you win, delete those two numbers; if you lose, append their sum as an extra term $x_{n+1} = x_1 + x_n$ at the right-hand end of the sequence. You play iteratively according to the above rule. If the sequence ever contains one term only, you wager that amount on an evens bet. If you win, you delete the term, and if you lose, you append it to the sequence to obtain two terms.
Show that, with probability 1, the game terminates with a profit of $\sum_{i=1}^n x_i$ and that the time until termination has finite mean.
This looks like another clever strategy. Show that the mean size of your largest stake before winning is infinite. (When Henry Labouchere was sent down from Trinity College, Cambridge, in 1852, his gambling debts exceeded £6000.)

3. (Maximalungleichungen und große Abweichungen).

a) Zeige: Ist M_n ein Martingal, dann gilt
$$P\left[\max_{k\leq n} M_k \geq c\right] \;\leq\; e^{-tc}\, E\left[e^{tM_n}\right] \qquad \forall\, c > 0.$$

b) Sei $S_n = Y_1 + Y_2 + \cdots + Y_n$ mit i.i.d. Zufallsvariablen $Y_i \in \mathcal{L}^1$, $E[Y_i] = 0$. Beweise folgende Erweiterung des Satzes von Chernoff:

$$P\left[\max_{k \leq n} S_k \geq a \cdot n\right] \;\leq\; e^{-\Lambda^*(a)\cdot n} \qquad \forall\, a > 0,$$

wobei $\Lambda^*(a) = \sup_{t>0}(ta - \Lambda(t))$, $\Lambda(t) = \log E[e^{tY_1}]$.

4. (Verzweigungsprozesse). Sei Z_n der Galton-Watson Verzweigungsprozess mit $Z_0 = 1$. Zeige:

a) Für $s \in [0,1]$ ist s^{Z_n} genau dann ein Martingal, wenn $G(s) = s$ gilt.

b) Im kritischen Fall ($m = 1$) stirbt der Prozess abgesehen von einem Ausnahmefall (welchem ?) P–fast sicher aus.

c) Die Wahrscheinlichkeit $\tilde{\pi} := P[M_\infty = 0]$ erfüllt $\tilde{\pi} = G(\tilde{\pi})$.

5. (Star trek II). 'Captain's Log...
Mr Spock and Chief Engineer Scott have modified the control system so that the Enterprise is confined to move for ever in a fixed plane passing through the sun. However, the next "hop-length" is now automatically set to be the current distance to the sun ("next" and "current" being updated in the obvious way). Spock is muttering something about logarithms and random walks, but I wonder whether it is (almost) certain that we will get into the solar system sometime...'

Hint: Let $X_n := \log R_n - \log R_{n-1}$. Prove that $X_1, X_2, \ldots$ is an i.i.d. sequence of variables, each of mean 0 and finite variance σ^2 (say), where $\sigma > 0$. Let

$$S_n := X_1 + X_2 + \cdots + X_n.$$

Prove that if α is a fixed positive number, then

$$\begin{aligned} P\left[\inf_n S_n = -\infty\right] &\geq P\left[S_n \leq -\alpha\sigma\sqrt{n}, \text{ infinitely often}\right] \\ &\geq \limsup_{n\to\infty} P\left[S_n \leq -\alpha\sigma\sqrt{n}\right] = \Phi(-\alpha) > 0. \end{aligned}$$