Stochastic analysis I WS 2007/08 Series 9

1. Solve the one-dimensional stochastic differential equation

$$dX_t \;=\; \left(\sqrt{1+X_t^2}+\frac{1}{2}X_t\right)dt \;+\; \sqrt{1+X_t^2}\,dB_t\;.$$

2. (Cameron–Martin Theorem). Let

$$X_t \;=\; B_t \,+\, h_t \qquad (0\le t\le 1)$$

where B_t is a Brownian motion on $(\Omega,\mathcal{A},P)$, and $h:[0,1]\to\mathbb{R}$ is an absolutely continuous function with $h'\in L^2([0,1],dt)$. *[From the point of view of stochastic differential equations X_t is the solution of the equation*

$$dX_t \;=\; dB_t \,+\, h'_t\,dt \tag{1}$$

with a deterministic drift, whereas from the point of view of analysis on Wiener space $X=(X_t)_{0\le t\le 1}$ is the displacement of the Brownian path by the constant (independent of ω) vector h]. Show :

a) For $0<s<t\le 1$ the Wiener-integrals $\int_0^s h'_u\,dB_u$ and $\int_s^t h'_u\,dB_u$ are independent and centered normally distributed with variance $\int_0^s (h'_u)^2\,du$ resp. $\int_s^t (h'_u)^2\,du$. In particular,

$$E_P\left[\exp\left(\int_s^t h'_u\,dB_u\right)\right] \;=\; \exp\left(\frac{1}{2}\int_s^t (h'_u)^2\,du\right)\;.$$

b) Conclude that

$$G_t \;:=\; \exp\left(-\int_0^t h'_u\,dB_u \;-\; \frac{1}{2}\int_0^t (h'_u)^2\,du\right)$$

is a martingale under P with $E_P[G_t]=1$.

c) Let Q be the probability measure on $(\Omega,\mathcal{A})$ that is absolutely continuous w.r.t. P with density $dQ/dP=G_1$. Show:

$$E_Q\left[\exp\left(\int_0^1 f_s\,dX_s\right)\right] \;=\; \exp\left(\frac{1}{2}\int_0^1 f_s^2\,ds\right)$$

for all $f \in L^2([0,1], dt)$. Compute the Fourier-transform

$$\varphi(p_1, \ldots, p_n) \;:=\; E_Q\left[\exp\left(i \sum_{j=1}^{n} p_j \cdot (X_{t_j} - X_{t_{j-1}})\right)\right]$$

of the distribution of $(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$, $0 = t_0 < t_1 < t_2 < \ldots t_n$ under Q.

Conclude, that the process $(X_t)_{0 \le t \le 1}$ is a Brownian motion under the probablility distribution Q.

Hence the change of measure from P to Q cancels the drift h'_t in the stochastic differential equation (1), resp. it undoes the displacement by h of the Brownian path.

d) Show that for bounded continuous $F : C([0,1]) \to \mathbb{R}$,

$$E_P[F(X)] \;=\; E_Q\left[F(X) \cdot \exp\left(\int_0^1 h'_s \, dX_s - \frac{1}{2}\int_0^1 (h'_s)^2 \, ds\right)\right].$$

Conclude that the distribution of X on $C([0,1])$ w.r.t. P is absolutely continuous w.r.t. Wiener measure with density

$$\exp\left(\int_0^1 h'_s \, dW_s - \frac{1}{2}\int_0^1 (h'_s)^2 \, ds\right),$$

where $W_s(\omega) := \omega(s)$ is the canonical Brownian motion on the Wiener space.

3. (Stochastic oscillator).

a) Let A and σ be $d \times d$–matrices, and B_t a Brownian motion in $\mathbb{R}^d$. Show that the unique solution of the stochastic differential equation

$$dZ_t \;=\; AZ_t \, dt \;+\; \sigma \, dB_t \,, \qquad Z_0 = z_0,$$

is given by

$$Z_t \;=\; e^{tA} Z_0 \;+\; \int_0^t e^{(t-s)A} \sigma \, dB_s.$$

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force are described by s.d.e. of type

$$\begin{aligned} dX_t &= V_t\, dt \\ dV_t &= -X_t\, dt \; + \; dB_t \end{aligned}$$

with a one-dimensional Brownian motion B_t (in complex notation: $dZ_t = -iZ_t\, dt \; + \; i\, dB_t$, $Z_t = X_t + iV_t$).
Solve the s.d.e. with initial condition $X_0 = x_0$, $V_0 = v_0$. Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation. Compute the limit

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{var}(X_t) \; .$$

4. (Martingale problem for Stratonovich s.d.e.). Let B_t be a Brownian motion in $\mathbb{R}^d$, and let X_t be a solution of the Stratonovich s.d.e.

$$\circ\, dX_t \;=\; b(X_t)\, dt \;+\; \sum_{k=1}^{d} \sigma_k(X_t) \circ dB_t^k, \qquad X_0 = x_0,$$

with coefficients $b \in C(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma_k \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. Show that X_t solves the martingale problem for the operator

$$\mathcal{L}f = \frac{1}{2} \sum_{k=1}^{d} \sigma_k \cdot \nabla(\sigma_k \cdot \nabla f) + b \cdot \nabla f, \qquad f \in C_0^2(\mathbb{R}^n).$$