Stochastic analysis WS 2007/08 Series 8

1. Solve the SDE

$$dX_t \;=\; -2\,\frac{X_t}{1-t}\,dt + \sqrt{2t(1-t)}\,dB_t, \qquad 0 \le t < 1,$$

with initial condition $X_0 = 0$. Show that the solution is a Gaussian process. Determine the covariance function $\mathrm{Cov}(X_s, X_t)$, and compare with that of the Brownian bridge.

2. (Change of time scale). Let $(B_t)_{t\ge 0}$ be a Brownian motion starting at 0.

a) Show that

$$X_t \;:=\; (1-t)\,B_{\frac{t}{1-t}} \quad \text{for } t<1\,, \qquad X_1 \;:=\; 0\,,$$

is a Brownian bridge from 0 to 0.

b) Derive this representation directly from the s.d.e. of the Brownian bridge.

3. (Lévy Area). If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\,ds \;=\; \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0,t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s\,dY_s \;-\; \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2}\,\left(X_t^2 + Y_t^2\right) \;+\; \beta(t)\,.$$

Show using Itô's formula, that e^{V_t} is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. The solution of the ordinary differential equations for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned} \alpha(t) &= p \cdot \tanh(p \cdot (t_0 - t)) \,, \\ \beta(t) &= -\log\cosh(p \cdot (t_0 - t)) \,. \end{aligned}$$

Conclude that

$$E\left[e^{ipA_{t_0}}\right] = \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R} \,.$$

c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) = \frac{1}{2t\cosh(\frac{\pi x}{2t})} \,.$$

4. (Estimation of Real-world σ's). If X_t is a stochastic process that models the price of a security at time t, then the random variable

$$R_k(h) = \frac{X_{kh}}{X_{(k-1)h}} - 1$$

is called the *k th period return.* It expresses in percentage terms the profit that one makes by holding the security from time $(k-1)h$ to time kh. When we use geometric Brownian motion

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t$$

as a model for the price of a security, one common misunderstanding is that σ can be interpreted as a normalized standard deviation of sample returns that more properly estimate

$$s = \sqrt{\mathrm{Var}(R_k(h))/h}.$$

Sort out this confusion by calculating $E[R_k(h)]$ and $\mathrm{Var}(R_k(h))$ in terms of μ and σ. Also, use these results to show that a honest formula for σ^2 is

$$\sigma^2 = \frac{1}{h} \log\left(1 + \frac{\mathrm{Var}(R_k(h))}{(1 + E[R_k(h)])^2}\right),$$

and suggest how you might estimate σ^2 from the data $R_1(h), R_2(h), \ldots, R_n(h)$.