Stochastic analysis I WS 2007/08 Series 7

1. (Variation of constants). The technique used for solving Exercise 6.1 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)\, dt + c(t) X_t\, dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions. Proceed as follows:

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz

$$X_t \;=\; C_t \cdot Z_t \;.$$

Show that the s.d.e. gets the form

$$\frac{dC_t(\omega)}{dt} \;=\; f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega)\,; \qquad C_0 = x. \tag{1}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (1) with ω as a parameter to find $C_t(\omega)$.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}\, dt + \alpha X_t\, dB_t; \qquad X_0 = x > 0,$$

where α is constant.

d) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma\, dt + \alpha X_t\, dB_t; \qquad X_0 = x > 0,$$

where α and γ are constants. For which values of γ do we get explosion?

2. (Complex Brownian motion). If (B_1, B_2) denotes a 2-dimensional Brownian motion we may introduce complex notation and put

$$\mathbf{B}(t) := B_1(t) + iB_2(t) \qquad (i = \sqrt{-1}).$$

$\mathbf{B}(t)$ is called *complex Brownian motion.*

a) If $F(z) = u(z) + iv(z)$ is an *analytic* function, and we define

$$Z_t = F(\mathbf{B}(t))$$

prove that

$$dZ_t = F'(\mathbf{B}(t))d\mathbf{B}(t), \tag{2}$$

where F' is the (complex) derivative of F. (Note that the usual second order terms in the (real) Itô formula are not present in (2)!)

b) Solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t \, d\mathbf{B}(t)$$

for constant α.

3. (Girsanov martingales and first order perturbations of the Laplace equation). Let $b \in C^1(D)$, and let u be a solution of the p.d.e.

$$\frac{1}{2}\Delta u + b \cdot \nabla u \;=\; 0 \quad \text{on } D, \qquad u = f \quad \text{on } \partial D.$$

a) Show that

$$M_t \;:=\; \exp\left(\int_0^t b(B_s) \cdot dB_s - \frac{1}{2}\int_0^t |b(B_s)|^2 ds\right), \qquad t < T_{D^c},$$

is a local martingale satisfying $dM_t = M_t \, b(B_t) \cdot dB_t$ and $d\langle M, B^i\rangle_t = M_t b^i(B_t) dt$.

b) By applying Itô's formula to the process $u(B_t)M_t$ prove that under appropriate assumptions,

$$u(x) \;=\; E_x[f(B_{T_{D^c}}) \, M_{T_{D^c}}].$$

(This is only a first example of the widespread use of exponential martingales in stochastic calculus – for more see any book on stochastic analysis (in particular Revuz/Yor) – keyword **Girsanov transform***).*