Stochastic analysis I WS 2007/08 Series 6

1. (Variation of constants). Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha\,Y_t\,dB_t$$

where r and α are real constants, and B_t is a one–dimensional Brownian motion.

2. (Feynman–Kac formula).

a) Let $D \subset \mathbb{R}^d$ be a bounded open domain, and let $V \in C(\bar{D})$ and $f \in C(\partial D)$. Suppose that $u \in C^2(D) \cap C(\bar{D})$ is a solution of the p.d.e.

$$\frac{1}{2}\Delta u - V\,u = 0 \quad \text{on } D, \qquad u = f \quad \text{on } \partial D.$$

Show that if $(B_t)_{t\geq 0}$ is a d–dimensional Brownian motion starting at $x \in D$ w.r.t. P_x, then

$$Z_t \;:= u(B_t) \cdot \exp\left(-\int_0^t V(B_s)\,ds\right), \qquad 0 \leq t \leq T_{D^c},$$

is a local martingale (w.r.t. which filtration?). Conclude that

$$u(x) \;=\; E_x\left[f(B_{T_{D^c}}) \cdot \exp\left(-\int_0^{T_{D^c}} V(B_s)\,ds\right)\right] \qquad \forall\, x \in D.$$

b) Similarly prove that under appropriate assumptions on V and f, a solution of the parabolic equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}\Delta u(t,x) + V(t,x)\,u(t,x) = 0$$

on $[0,t_0] \times \mathbb{R}^d$ with boundary condition $u(t_0,x) = f(x)$ satisfies the *Feynman–Kac formula*

$$u(t,x) = E_x\left[f(B_{t_0-t}) \cdot \exp\left(-\int_t^{t_0} V(s,B_{s-t})\,ds\right)\right].$$

(A corresponding representation for solutions of the time-dependent Schrödinger equation has been derived heuristically by Feynman – however in Feynman's approach, integration w.r.t Wiener measure is replaced by integration w.r.t a (not mathematically well-defined) infinite-dimensional Lebesgue measure. For more background on Feynman path integrals and their application in quantum physics see the Feynman lectures or the first chapter in Glimm/Jaffe: Quantum physics.)

3. (Polar sets for Brownian motion). Let B_t be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$, with start in x_0.

a) Show that for $d = 2$ every point $x \in \mathbb{R}^d$ is polar, i.e.,

$$P\,[\,\exists\, t > 0 \,:\, B_t = x\,] \;=\; 0.$$

Hint: First consider the case $x \neq x_0$.

b) Conclude that in general, $d - 2$ dimensional subspaces are polar for Brownian paths.

4. (Covariation of Itô processes). Compute the covariation of two Itô processes

$$I_t = \int_0^t G_s\, dB_s^1 \qquad \text{and} \qquad J_t = \int_0^t H_s\, dB_s^2\,,$$

where B_t^1 and B_t^2 are independent $(\mathcal{F}_t)$–Brownian motions, and G_t and H_t are continuous $(\mathcal{F}_t)$–adapted processes.