Stochastic analysis I WS 2007/08 Series 5

1. (Solutions of stochastic differential equations).

a) Verify that the given processes solve the corresponding stochastic differential equations (B_t denotes a one–dimensional Brownian motion):

i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2} X_t \, dt + X_t \, dB_t$.

ii) $X_t = \frac{B_t}{1+t}$; $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} \, dB_t; \qquad X_0 = 0.$$

iii) $X_t = \sin B_t$ with $B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solves

$$dX_t = -\frac{1}{2} X_t \, dt + \sqrt{1 - X_t^2} \, dB_t \quad \text{for } t < \inf \left\{ s > 0; B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}.$$

iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t.$$

v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t.$$

b) Show that the process $X_t = (B_t + t) \exp(-B_t - \frac{1}{2} t)$ is a martingale.

2. (Geometric Brownian motion). A geometric brownian motion with parameters $\mu, \alpha \in \mathbb{R}$ is a solution of the stochastic differential equation

$$dX_t \;=\; \mu\, X_t\, dt \,+\, \alpha\, X_t\, dB_t\,.$$

It is used for instance in the modelling of stock prices.

a) Find a solution of the SDE starting in $X_0 = x_0$ by using the ansatz

$$X_t \;=\; x_0 \cdot e^{aB_t + bt}\,.$$

b) Calculate $E[X_t]$ for $t \geq 0$. What is striking in the case $0 < \mu < \alpha^2/2$? Calculate $\mathrm{cov}\,(X_s, X_t)$.

3. (Multidimensional Itô formula).

a) State and prove Itô's formula for $\mathbb{R}^d$-valued processes.

b) Deduce the Itô formula for time-dependent processes in $\mathbb{R}^d$.

4. (Stochastic representation of solutions of the Poisson problem). Let $D \subset \mathbb{R}^d$ be a bounded domain. Show that if $u \in C(\bar{D}) \cap C^2(D)$ is a solution of the Poisson problem

$$\frac{1}{2}\,\Delta u \;=\; -g \;\text{ in } D\,, \qquad u = f \;\text{ in } \partial D\,,$$

then

$$u(x) \;=\; E_x\left[\int_0^{T_D} g(B_t)\, dt\right] \,+\, E_x\left[f(B_{T_D})\right] \qquad \forall\, x \in D\,,$$

where B_t is a Brownian motion starting in x with respect to P_x.