Stochastic analysis I WS 2007/08 Series 4

1. (Stratonovich calculus). Let $X : [0,\infty) \to \mathbb{R}$ be continuous with continuous quadratic variation (with respect to a fixed sequence of partitions τ_n with $|\tau_n| \to 0$). Show that for $F \in C^2$ the pathwise Stratonovich integral

$$\int_0^t F'(X_s) \circ dX_s \ := \ \lim_{n\to\infty} \sum_{\substack{s\in\tau_n \\ s<t}} \frac{1}{2} \left(F'(X_{s'\wedge t}) + F'(X_s)\right) \cdot (X_{s'\wedge t} - X_s)$$

exists, and the chain rule

$$F(X_t) - F(X_0) \ = \ \int_0^t F'(X_s) \circ dX_s$$

holds. Why does one not consider Stratonovich instead of Itô integrals in general ?

2. (Moments of Brownian motion). Consider a one-dimensional Brownian motion B_t with start in 0.

a) Use Itô's formula to prove that

$$E[B_t^k] = \frac{1}{2}k(k-1)\int_0^t E[B_s^{k-2}]ds \qquad \text{for all } k \geq 2.$$

b) Conclude that

$$E[B_t^{2k+1}] = 0$$

and

$$E[B_t^{2k}] = \frac{(2k)!t^k}{2^k k!} \ .$$

3. (Extended Itô formula). Let $X : [0, \infty) \to \mathbb{R}$ be continuous with continuous quadratic variation w.r.t. (τ_n). Show that for $g \in C^1$ and $F \in C^2$ the pathwise Itô integral

$$\int_0^t g(X_s) dF(X_s)$$

exists, and Itô's formula

$$\int_0^t g(X_s) dF(X_s) = \int_0^t g(X_s) F'(X_s) dX_s + \frac{1}{2} \int_0^t g(X_s) F''(X_s) d\langle X \rangle_s$$

holds. This justifies the differential notation

$$dF(X) = F'(X) dX + \frac{1}{2} F'' d\langle X \rangle.$$

4. (Quadratic variation of Itô integrals). Let $X : [0, \infty) \to \mathbb{R}$ be a continuous function with continuous quadratic variation.

a) Let $F \in C^1(\mathbb{R})$. Show that the function $t \mapsto F(X_t)$ has quadratic variation

$$\langle F(X) \rangle_t = \int_0^t F'(X_s)^2 d\langle X \rangle_s \,.$$

b) Conclude that for $f \in C^1(\mathbb{R})$ the Itô integral

$$I_t(f) = \int_0^t f(X_s) dX_s$$

has quadratic variation

$$\langle I(f) \rangle_t = \int_0^t f(X_s)^2 \, d\langle X \rangle_s.$$