Stochastic analysis WS 2007/08 Series 3

1. (Wiener integrals). Consider the stochastic integral

$$I_t \;:=\; \int_0^t h_s \, dB_s, \qquad 0 \le s \le 1,$$

of a *deterministic* integrand $h \in L^2([0,1], ds)$ w.r.t. Brownian motion.

a) Give a Riemann sum approximation of the integral, and use this to prove that for all $t \in [0,1]$, I_t is normally distributed with mean zero and variance

$$\tau(t) \;=\; \int_0^t h_r^2 \, dr.$$

b) Show more generally that I_t is a continuous Gaussian process with mean zero and covariance

$$\mathrm{Cov}\,(I_s, I_t) \;=\; \int_0^{s\wedge t} h_r^2 \, dr.$$

c) Conclude that $(I_t)_{0\le t\le 1}$ has the same distribution on $C([0,1])$ as the time changed Brownian motion

$$t \mapsto B_{\tau(t)}, \qquad 0 \le t \le 1.$$

2. (Integration w.r.t. an Itô process). Let

$$I_s \; := \; \int_0^s H_u \, dB_u, \qquad 0 \le s \le t,$$

with an $(\mathcal{F}_s)$–Brownian motion B_s on $(\Omega, \mathcal{A}, P)$, and an $(\mathcal{F}_s)$–adapted process $H \in L^2(P \otimes \lambda)$. Suppose that τ_n is a sequence of partitions of $[0, t]$ such that $|\tau_n| \to 0$. Prove that if $(G_s)_{0 \le s \le t}$ is another $(\mathcal{F}_s)$–adapted continuous bounded process, then the Riemann sums $\sum_{s \in \tau_n} G_s \cdot (I_{s'} - I_s)$ converge in $L^2(P)$, and

$$\int_0^t G_s \, dI_s \; := \; \lim_{n \to \infty} \sum_{s \in \tau_n} G_s \cdot (I_{s'} - I_s) \; = \; \int_0^t G_s \, H_s \, dB_s \quad P\text{–a.s.}$$

Hint: Express the Riemann sums as a stochastic integral $\int_0^t \ldots dB_s$ w.r.t. Brownian motion).

3. (A local martingale that is not a martingale). Let B_t $(t \ge 0)$ be a Brownian motion in $\mathbb{R}^3$ with start in $x \neq 0$. Show:

a) $X_t = 1/\|B_t\|$ is a local martingale up to $T = \infty$ w.r.t. the filtration $\mathcal{F}_t$ generated by B_t.

b) $\{X_s | \, 0 \le s \le t\}$ is uniformly integrable for all $t \ge 0$.

c) X_t is *not* a martingale.

4. (Stopping Theorem). State the stopping theorem for a continuous martingale $(M_t)_{t \ge 0}$. Give a complete detailed proof.