Stochastic analysis WS 2007/08 Series 2

1. (Convergence of Riemann-Itô sums).

a) Let H_t be an adapted, product measurable process that is *continuous in mean square*, i.e.

$$\lim_{s\to t} E\left[(H_s - H_t)^2\right] \;=\; 0 \qquad \text{for all } t \geq 0.$$

Show that

$$\int_0^t H_s\,dB_s \;=\; \lim_{n\to\infty} \sum_{s\in\tau_n} H_s \cdot (B_{s'} - B_s) \qquad \text{in } L^2(P)$$

for any sequence τ_n of partitions of $[0,t]$ such that $|\tau_n| \to 0$.

b) Show that $H_t = f(B_t)$ is continuous in mean square, if B_t is a Brownian motion, and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

2. (Quadratic variation of Brownian motion). Prove that

$$\lim_{n\to\infty} \sum_{s\in\tau_n} (B_{s'} - B_s)^2 \;=\; t \qquad \text{in } L^2(P)$$

for any sequence τ_n of partitions of $[0,t]$ such that $|\tau_n| \to 0$.

Hint: Compute $E[\sum((B_{s'} - B_s)^2 - (s' - s))^2]$.

3. (Itô and Stratonovich formula for B_t^2).

a) Prove directly from the definition of the Itô integral that

$$\int_0^t B_s\,dB_s \;=\; \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

The assertion of Exercise 2 can be assumed.

b) The Stratonovich integral of B_t w.r.t. B_t is defined by

$$\int_0^t B_s \circ dB_s \;:=\; \lim_{n\to\infty} \sum_{s\in\tau_n} \frac{1}{2}(B_{s'} + B_s)\cdot(B_{s'} - B_s) \qquad \text{in } L^2(P).$$

Show that

$$\int_0^t B_s \circ dB_s \;=\; \frac{1}{2} B_t^2 \,,$$

so the Itô and Stratonovich integrals do not coincide !

4. (Maximal inequality). State Doob's $\mathcal{L}^p$ inequality for a continuous martingale $(M_t)_{t\geq 0}$. Give a complete detailed proof.