Stochastic analysis I WS 2007/08 Series 13

1. (Girsanov theorem for local martingales). Let M_t and L_t be local martingales with respect to P such that

$$Z_t \;:=\; \exp\left(L_t - \frac{1}{2}\langle L\rangle_t\right)$$

is a martingale with respect to P, and consider the probability measure $Q \overset{\text{loc}}{\ll} P$ with local densities Z_t.

a) Prove that
$$\tilde{M}_t \;:=\; M_t - \langle M, L\rangle_t$$
is a local martingale w.r.t. Q with covariations
$$\langle \tilde{M}, N\rangle_t \;=\; \langle M, N\rangle_t$$
for any local martingale N.

b) Conclude that a semimartingale w.r.t. P is also a semimartingale w.r.t. Q.

2. (A connection between Brownian motion with drift and Brownian motion with absorption). Let B_t be a d-dimensional Brownian motion with start in x w.r.t. P_x, and suppose that X_t is a solution of the s.d.e

$$dX_t \;=\; \nabla h(X_t)\,dt \,+\, dB_t\,, \qquad X_0 = x\,, \tag{1}$$

with $h \in C_0^2(\mathbb{R}^d)$.

a) Show that for $f \in C_0(\mathbb{R}^d)$
$$E_x[f(X_t)] \;=\; E_x\left[e^{-\int_0^t V(B_s)\,ds}\, e^{h(B_t)-h(x)}\, f(B_t)\right]\,,$$
where
$$V(x) \;:=\; \frac{1}{2}\,|\nabla h(x)|^2 \,+\, \frac{1}{2}\Delta h(x)\,.$$
(Uniqueness in distribution of (1) can be assumed without proof).

b) Conclude that for $V \geq 0$, the transition functions $(p_t f)(x) := E_x[f(X_t)]$ of Brownian motion with drift ∇h satisfy

$$p_t f \;=\; e^{-h}\, p_t^V (e^h f)\,,$$

where p_t^V are the transition functions of Brownian motion with absorption rate $V(x)$.

Remark: The generator of p_t is $\mathcal{L} = \frac{1}{2}\Delta + \nabla h \cdot \nabla$, whereas the generator of p_t^V is the Schrödinger operator $\mathcal{L}^V = \frac{1}{2}\Delta + V$. The ground state transform of quantum mechanics transforms the Schrödinger operator $\mathcal{L}^V$ into $\mathcal{L}$. Hence the transformation ,,BM with drift $\mapsto$ BM with absorption" *can be viewed as a stochastic version of an inverse ground state transform.*

3. (Feynman–Kac formula and stock prices). State without proof the time-dependent Feynman–Kac formula. Now consider a stock price that is described by a geometric Brownian motion X_t with parameters $\alpha, \mu > 0$. Suppose that at price x expenses $V(x)$ accrue — the total cost at time t then is

$$A_t \;=\; \int_0^t V(X_s)\, ds\ .$$

a) Derive a partial differential equation for the Laplace transform

$$u(t,x) \;=\; E_x\left[e^{-\beta A_t}\right] \qquad (\beta > 0)$$

of A_t from the Feynman-Kac formula for Brownian motion.

b) Show that X_t solves the time-dependent martingale problem for the generator

$$\mathcal{L} \;=\; \frac{\alpha^2}{2}\, x^2\, \frac{d^2}{dx^2} \;+\; \mu\, x\, \frac{d}{dx}.$$

Conclude that under appropriate conditions $u(t,x)$ also solves the partial differential equation

$$\frac{\partial u}{\partial t} \;=\; \mathcal{L}u \;-\; \beta V u.$$

4. (Support of Wiener measure). Show that the support of Wiener measure on $C([0,1])$ is the whole space.