Stochastic analysis I WS 2007/08 Series 12

1. (Brownian motion writes your name). Prove that Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted i's or crossed t's).
To get the pen rolling, first take B_t to be a two-dimensional Brownian motion on $[0,1]$, and note that for any $[a,b] \subset [0,1]$ the process

$$X_t^{(a,b)} = (b-a)^{1/2}(B_{a+t/(b-a)} - B_a)$$

is again a Brownian motion on $[0,1]$. Now, take $g : [0,1] \to \mathbb{R}^2$ to be a parametrization of your name, and note that Brownian motion spells your name (to precision ϵ) on the interval (a,b) if

$$\sup_{0\le t\le 1} |X_t^{a,b} - g(t)| \le \epsilon. \tag{1}$$

a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that A_k are independent events and that one has $P(A_k) = P(A_1)$ for all k. Next, use the Borel-Cantelli lemma to show that if $P(A_1) > 0$ then infinitely many of the A_k will occur with probability 1.

b) Consider an extremely dull individual whose signature is maximally undistinguished so that $g(t) = (0,0)$ for all $t \in [0,1]$. This poor soul does not even make an X; his signature is just a dot. Show that

$$P\left(\sup_{0\le t\le 1} |B_t| \le \epsilon\right) > 0. \tag{2}$$

b) Finally, complete the solution of the problem by using (2) and an appropriate Girsanov theorem to show that $P(A_1 > 0)$; that is to prove

$$P\left(\sup_{0\le t\le 1} |B_t - g(t)| \le \epsilon\right) > 0.$$

2. (Concentration of measure). Let M be a continuous local martingale satisfying $M_0 = 0$. Show that

$$P\left[\max_{s\le t} M_s \ge y\,,\, \langle M\rangle_t \le K\right] \;\le\; \exp\left(-\frac{y^2}{2K}\right) \quad \forall\, t, y, K > 0\,.$$

3. (Drift-transformation by change of measure). Let $\beta : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathrm{Hom}\,(\mathbb{R}^d, \mathbb{R}^n)$ be product-measurable and adapted, and let

$$b(t, x) \ := \ \sigma(t, x)\, \beta(t, x) \ .$$

Show that if under P, X_t is a solution of the s.d.e.

$$dX_t \ = \ \sigma(t, X_t)\, dB_t \ ,$$

with an $\mathbb{R}^d$ valued Brownian motion B_t, and

$$Z_t \ := \ \exp\left(\int_0^t \beta(s, X_s)\, dB_s - \int_0^t |\beta(s, X_s)|^2\, ds \right)$$

is a martingale, then X_t solves the s.d.e.

$$dX_t \ = \ b(t, X_t)\, dt \ + \ \sigma(t, X_t)\, dW_t \ ,$$

where W_t is a Brownian motion w.r.t. the transformed measure with local densities Z_t.

4. (Stopping times). Let S and T be $(\mathcal{F}_t)$-stopping times, and $X, Y \in \mathcal{L}^1$. Prove:

a) If $X = Y$ P-a.s. on $A \in \mathcal{F}_S$, then

$$E_P[X|\mathcal{F}_S] = E_P[Y|\mathcal{F}_S] \qquad P\text{-a.s. on } A$$

b)

$$\begin{aligned} E_P\left[E_P[X|F_T]|F_S\right] \ &= \ E_P[X|F_{T\wedge S}] \qquad P\text{-a.s., } \textit{and} \\ E_P\left[E_P[X|F_T]|F_S\right] \ &= \ E_P[X|F_S] \qquad P\text{-a.s. on } \{S \le T\} \end{aligned}$$