Stochastic analysis I WS 2007/08 Series 10 (for Monday 14.1.)

1. (Lévy's characterization of Brownian motion). Show via Levy's theorem:

a) If $(B_t)_{t\geq 0}$ is a Brownian motion and $c > 0$, then $\left(\sqrt{c}B_{\frac{t}{c}}\right)_{t\geq 0}$ is also a Brownian motion.

b) If $(B_t^i)_{t\geq 0}$ $(i = 1, \dots, n)$ are independent Brownian motions, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} B_t^i \quad (t \geq 0)$$

is a Brownian motion.

2. (Covariation of local martingales). Let M and N be local martingales. Show that $\langle M, N\rangle$ is the P-a.s. unique continuous process of bounded variation with $\langle M, N\rangle_0 = 0$ such that

$$M_t N_t - \langle M, N\rangle_t$$

is a local martingale.

3. (Quadratic variation). Let $(X_t)_{t\geq 0}$ be a continuous martingale, and let S and T be stopping times satisfying $S \leq T$. Show:

$$\langle X\rangle_S = \langle X\rangle_T \;\Rightarrow\; X \text{ is a.s. constant on } [S, T].$$

4. (Hitting time of an open set). Let $(M_t)_{t\geq 0}$ be a continuous local $(\mathcal{F}_t)$-martingale with $M_0 = 0$. Prove that the first hitting time

$$T_s \;:=\; \inf\{t \geq 0 : \langle M\rangle_t > s\}$$

is an $(\mathcal{F}_t)$-stopping time if $(\mathcal{F}_t)$ is right-continuous.

Series 11 (for Monday 21.1.)

1. (Covariance of stochastic integrals). Let M and N be square integrable continuous martingales with absolutely continuous quadratic variations. Prove the following extension of the Itô isometry

a) for elementary adapted integrands G, H,

b) for all $G \in \mathcal{L}_a^2(P_M)$ and $H \in \mathcal{L}_a^2(P_N)$:

$$E\left[\int_s^t G\,dM \int_s^t H\,dN \,\middle|\, \mathcal{F}_s\right] \;=\; E\left[\int_s^t GH\,d\langle M,N\rangle \,\middle|\, \mathcal{F}_s\right] \quad \forall\, 0 \le s \le t.$$

2. (Stopped Brownian motion). Prove that a continuous local martingale M with quadratic variation $\langle M\rangle_t = t \wedge T$, where T is a stopping time, is a Brownian motion stopped at T.

3. (Quadratic variation of Poisson processes). Let N_t be a Poisson process of intensity 1. Show that the compensated Poisson process

$$M_t \;:=\; N_t - t$$

is a martingale, and that $M_t^2 - t$ is a martingale as well. Thus $A_t := t$ is the increasing process in the Doob-Meyer decomposition of M_t^2. Can A_t be interpreted as a quadratic variation along a sequence (τ_n) of partitions with $|\tau_n| \to 0$?

4. (A "counterexample" to Lévy's theorem ?).
Let $W_t = (W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$ be a three-dimensional Brownian motion starting at the origin, and define

$$X = \prod_{i=1}^{3} \operatorname{sgn}(W_1^{(i)}) \;,$$

$$M_t^{(1)} = W_t^{(1)}, \quad M_t^{(2)} = W_t^{(2)}, \quad \text{and} \quad M_t^{(3)} = XW_t^{(3)}.$$

Show that each of the pairs $(M^{(1)}, M^{(2)})$, $(M^{(1)}, M^{(3)})$, $(M^{(2)}, M^{(3)})$ is a two-dimensional Brownian motion, but $(M_t^{(1)}, M_t^{(2)}, M_t^{(3)})$ is *not* a three-dimensional Brownian motion. Explain why this does not provide a counterexample to Lévy's theorem.