Stochastic analysis I WS 2007/08 Series 1

TUTORIAL CLASSES: Mondays 12-2, SR 501, We. 6

1. (Stieltjes–calculus for functions of bounded variation).

a) Define the Lebesgue–Stieltjes–integral

$$\int_0^t f(s)\,dg(s)$$

of a locally bounded Borel-measurable function $f : [0,\infty) \to \mathbb{R}$ w.r.t an increasing continuous function $g : [0,\infty) \to \mathbb{R}$.

b) For a continuous function $g : [0,\infty) \to \mathbb{R}$ of bounded variation let

$$V_t^{(1)} \;:=\; \sup_\tau \sum_i |g(t_{i+1}) - g(t_i)|\ ,$$

where the supremum is taken over all partitions of $[0,t]$. Show that $V^{(1)}$ and $V^{(1)} - g$ are increasing. Use this result to extend the definition of the Lebesgue–Stieltjes integral to functions of bounded variation.

c) Let τ_n be a sequence of partitions satisfying $|\tau_n| \to 0$. Prove: When g is continuous and of bounded variation and f is continuous, then the integral

$$\int_0^t g(s)\,df(s) \;:=\; \lim_{n\to\infty} \sum_{\substack{t_i\in\tau_n\\ t_i\le t}} g(t_i)\,(f(t_{i+1}) - f(t_i))$$

exists, and the integration by parts identity

$$\int_0^t f(s)\,dg(s) \;=\; f(t)g(t) \;-\; f(0)g(0) \;-\; \int_0^t g(s)\,df(s)$$

holds. In particular, $\int g\,df$ is independent of the choice of the sequence $(\tau_n)_{n\in\mathbb{N}}$.

2. (Wiener–Paley definiton of stochastic integrals). Let $(B_t)_{t\ge0}$ be a one-dimensional Brownian motion with $B_0 = 0$. For a continuous function $h : [0,1] \to \mathbb{R}$ of bounded variation the integral $\int_0^1 h(s)\,dB_s$ can be defined P–a.s. as in Exercise 1 c). Prove:

$$E\left[\int_0^1 h(s)\,dB_s\right] \;=\; 0$$

and

$$E\left[\left(\int_0^1 h(s)\,dB_s\right)^2\right] \;=\; \int_0^1 (h(s))^2\;ds\,.$$

Use this isometry to define the integral $\int_0^1 h(s)dB_s$ for all functions $h \in L^2(0,1)$.

3. (Discrete stochastic differential equations). Let $Y_1, Y_2, \ldots \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$ be i.i.d. random variables with $E[Y_i] = 0$ and $\mathrm{Var}(Y_i) = 1$. For fixed $h > 0$ let $\Pi_h := \{0, h, 2h, \ldots\}$. The process $X_t^{(h)} : \Omega \to \mathbb{R}$ is defined by

$$\begin{aligned} X_0^{(h)} &= x_0 \\ X_{(k+1)h}^{(h)} - X_{kh}^{(h)} &= b(X_{kh}^{(h)})\,h + \sigma(X_{kh}^{(h)})\sqrt{h}\,Y_{k+1}. \end{aligned}$$

Let $\mathcal{F}_{nh} := \sigma(Y_1, \ldots, Y_n)$ be the filtration generated by the random variables Y_i. Prove:

a) $(X_t^{(h)})_{t\in\Pi_h}$ is a *Markov chain* with respect to $(\mathcal{F}_t)_{t\geq 0}$ with transition probabilities

$$p(x,\cdot) = P \circ \left(x + b(x)\,h + \sigma(x)\sqrt{h}\,Y_1\right)^{-1}.$$

b) The Doob decomposition of $X_t^{(h)}$ is given by $X_t^{(h)} = A_t^{(h)} + M_t^{(h)}$ where

$$\begin{aligned} A_{nh}^{(h)} &= x_0 + \sum_{k=0}^{n-1} b(X_{kh}^{(h)})\,h \qquad \text{, and} \\ M_{nh}^{(h)} &= \sum_{k=0}^{n-1} \sigma(X_{kh}^{(h)})\sqrt{h}\,Y_{k+1}. \end{aligned}$$

c) Show that for $t \in \Pi_h$,

$$\begin{aligned} \mathrm{Var}(M_{t+h}^{(h)} - M_t^{(h)}|\mathcal{F}_t) &= \sigma(X_t^{(h)})^2\,h \\ \mathrm{Var}(M_t^{(h)}) &= \sum_{s<t,s\in\Pi_h} E[\sigma(X_s^{(h)})^2]h. \end{aligned}$$