A1

9. By integration, for $x, y > 0$,

$$f_Y(y) = \int_0^y f(x,y)\,dx = \tfrac{1}{6}cy^3e^{-y}, \quad f_X(x) = \int_x^\infty f(x,y)\,dy = cxe^{-x},$$

whence $c = 1$. It is simple to check the values of $f_{X|Y}(x \mid y) = f(x,y)/f_Y(y)$ and $f_{Y|X}(y \mid x)$, and then deduce by integration that $\mathbb{E}(X \mid Y = y) = \frac{1}{2}Y$ and $\mathbb{E}(Y \mid X = x) = x + 2$.

A2

5. The density function of $X_1 + X_2$ is, by convolution,

$$f_2(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 2 - x & \text{if } 1 \le x \le 2. \end{cases}$$

Therefore, for $1 \le x \le 2$,

$$f_3(x) = \int_0^1 f_2(x-y)\,dy = \int_{x-1}^1 (x-y)\,dy + \int_0^{x-1} (2-x+y)\,dy = \tfrac{3}{4} - (x - \tfrac{3}{2})^2.$$

Likewise,

$$f_3(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \le x \le 1, \\ \frac{1}{2}(3-x)^2 & \text{if } 2 \le x \le 3. \end{cases}$$

A simple induction yields the last part.

A3

6. The covariance satisfies $\text{cov}(U, V) = \mathbb{E}(X^2 - Y^2) = 0$, as required. If X and Y are symmetric random variables taking values ± 1, then

$$\mathbb{P}(U = 2,\ V = 2) = 0 \quad \text{but} \quad \mathbb{P}(U = 2)\mathbb{P}(V = 2) > 0.$$

If X and Y are independent $N(0,1)$ variables, $f_{U,V}(u,v) = (4\pi)^{-1}e^{-\frac{1}{4}(u^2+v^2)}$, which factorizes as a function of u multiplied by a function of v.

A4

13. The random variable X^{-1} is symmetric and, for $a > 0$,

$$\mathbb{P}(X^{-1} > a) = \mathbb{P}(0 < X < a^{-1}) = \int_0^{a^{-1}} \frac{du}{\pi(1+u^2)} = \int_\infty^a \frac{-v^{-2}\,dv}{\pi(1+v^{-2})},$$

by the transformation $v = 1/u$. For another example, consider the density function

$$f(x) = \begin{cases} \frac{1}{2}x^{-2} & \text{if } x > 1, \\ \frac{1}{2} & \text{if } 0 \le x \le 1. \end{cases}$$

A6

14. The transformation $w = x + y$, $z = x/(x+y)$ has inverse $x = wz$, $y = (1-z)w$, and Jacobian $J = w$, whence

$$\begin{aligned} f(w,z) &= w \cdot \frac{\lambda(\lambda wz)^{\alpha-1}e^{-\lambda wz}}{\Gamma(\alpha)} \cdot \frac{\lambda(\lambda(1-z)w)^{\beta-1}e^{-\lambda(1-z)w}}{\Gamma(\beta)} \\ &= \frac{\lambda(\lambda w)^{\alpha+\beta-1}e^{-\lambda w}}{\Gamma(\alpha+\beta)} \cdot \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)}, \qquad w > 0,\ 0 < z < 1. \end{aligned}$$

Hence W and Z are independent, and Z is beta distributed with parameters α and β.

A7

6. We confine ourselves to the more interesting case when $\rho \ne 1$. Writing $X = U$, $Y = \rho U + \sqrt{1-\rho^2}V$, we have that U and V are independent $N(0,1)$ variables. It is easy to check that $Y > X$ if and only if $(1-\rho)U < \sqrt{1-\rho^2}V$. Turning to polar coordinates,

$$\mathbb{E}(\max\{X,Y\}) = \int_0^\infty \frac{re^{-\frac{1}{2}r^2}}{2\pi}\left[\int_\psi^{\psi+\pi}\left\{\rho r\cos\theta + r\sqrt{1-\rho^2}\sin\theta\right\}d\theta + \int_{\psi-\pi}^{\psi} r\cos\theta\,d\theta\right]dr$$

where $\tan\psi = \sqrt{(1-\rho)/(1+\rho)}$. Some algebra yields the result. For the second part,

$$\mathbb{E}(\max\{X,Y\}^2) = \mathbb{E}(X^2 I_{\{X>Y\}}) + \mathbb{E}(Y^2 I_{\{Y>X\}}) = \mathbb{E}(X^2 I_{\{X<Y\}}) + \mathbb{E}(Y^2 I_{\{Y<X\}}),$$

by the symmetry of the marginals of X and Y. Adding, we obtain $2\mathbb{E}(\max\{X,Y\}^2) = \mathbb{E}(X^2) + \mathbb{E}(Y^2) = 2$.

A 8

3. Either integrate by parts or use Fubini's theorem:

$$r\int_0^\infty x^{r-1}\mathbb{P}(X>x)\,dx = r\int_0^\infty x^{r-1}\left\{\int_{y=x}^\infty f(y)\,dy\right\}dx$$
$$=\int_{y=0}^\infty f(y)\left\{\int_{x=0}^{y} rx^{r-1}\,dx\right\}dy=\int_0^\infty y^r f(y)\,dy.$$

An alternative proof is as follows. Let I_x be the indicator of the event that $X>x$, so that $\int_0^\infty I_x\,dx = X$. Taking expectations, and taking a minor liberty with the integral which may be made rigorous, we obtain $\mathbb{E}X=\int_0^\infty \mathbb{E}(I_x)\,dx$. A similar argument may be used for the more general case.

A 9

5. It is a standard to write $X=X^+-X^-$ where $X^+=\max\{X,0\}$ and $X^-=-\min\{X,0\}$. Now X^+ and X^- are non-negative, and so, by Lemma (4.3.4),

$$\mu=\mathbb{E}(X)=\mathbb{E}(X^+)-\mathbb{E}(X^-)=\int_0^\infty \mathbb{P}(X>x)\,dx-\int_0^\infty \mathbb{P}(X<-x)\,dx$$
$$=\int_0^\infty [1-F(x)]\,dx-\int_0^\infty F(-x)\,dx=\int_0^\infty [1-F(x)]\,dx-\int_{-\infty}^0 F(x)\,dx.$$

It is a triviality that

$$\mu=\int_0^\mu F(x)\,dx+\int_0^\mu [1-F(x)]\,dx$$

and the equation follows with $a=\mu$. It is easy to see that it cannot hold with any other value of a, since both sides are monotonic functions of a.

A 10

3. If g is strictly decreasing then $\mathbb{P}(g(X)\le y)=\mathbb{P}(X\ge g^{-1}(y))=1-g^{-1}(y)$ so long as $0\le g^{-1}(y)\le 1$. Therefore $\mathbb{P}(g(X)\le y)=1-e^{-y}$, $y\ge 0$, if and only if $g^{-1}(y)=e^{-y}$, which is to say that $g(x)=-\log x$ for $0<x<1$.

A 11

8. One way is to evaluate

$$\int_0^\infty\int_x^\infty\int_y^\infty \lambda\mu\nu e^{-\lambda x-\mu y-\nu z}\,dx\,dy\,dz.$$

Another way is to observe that $\min\{Y,Z\}$ is exponentially distributed with parameter $\mu+\nu$, whence $\mathbb{P}(X<\min\{Y,Z\})=\lambda/(\lambda+\mu+\nu)$. Similarly, $\mathbb{P}(Y<Z)=\mu/(\mu+\nu)$, and the product of these two terms is the required answer.

A 12

8. By definition, $\mathbb{E}(e^{itX})=\mathbb{E}(\cos(tX))+i\mathbb{E}(\sin(tX))$. By integrating by parts,

$$\int_0^\infty \cos(tx)\lambda e^{-\lambda x}\,dx=\frac{\lambda^2}{\lambda^2+t^2},\qquad \int_0^\infty \sin(tx)\lambda e^{-\lambda x}\,dx=\frac{\lambda t}{\lambda^2+t^2},$$

and

$$\frac{\lambda^2+i\lambda t}{\lambda^2+t^2}=\frac{\lambda}{\lambda-it}.$$