Reflection coupling and Wasserstein contractivity without convexity

Andreas Eberle

September 3, 2011

1 INTRODUCTION

$$dX_t \;=\; dB_t \,+\, b(X_t)\,dt\,, \qquad b:\mathbb{R}^d \to \mathbb{R}^d \text{ locally Lipschitz.}$$

GOAL :

- Exponential contractivity of transition kernel $p_t(x,dy)$ w.r.t. Kantorovich-Rubinstein (L^1 Wasserstein) distance

$$\mathcal{W}_f(\mu,\nu) \;=\; \inf_{c\in\Pi(\mu,\nu)} \int f(|x-y|)\;c(dx\,dy)\,,$$

  $f:\mathbb{R}^+ \to \mathbb{R}^+$ strictly increasing, concave, $f(0)=0$.

- Optimization of exponential decay rate by appropriate choice of f.
  - *Minimal concavity:* $f(x)=x \;\Rightarrow\; \mathcal{W}_f =$ standard KRW distance
  - *Maximal concavity:* $f(x)=I_{\{x\neq 0\}} \;\Rightarrow\; \mathcal{W}_f =$ TV distance

  The optimal choice is inbetween !

REFLECTION COUPLING (Lindvall and Rogers 1986)

$$\begin{aligned} dX_t &= b(X_t)\,dt + dB_t\,, \qquad X_0 \sim \mu, \\ dY_t &= b(Y_t)\,dt + (I - 2e_t e_t^T)\,dB_t\,, \qquad Y_0 \sim \nu, \end{aligned}$$

for $t < T = \inf\{t \geq 0 : X_t = Y_t\}$, $Y_t = X_t$ for $t \geq T$, where

$$e_t := Z_t/|Z_t|, \qquad Z_t := X_t - Y_t.$$

Equation for difference vector:

$$dZ_t = (b(X_t) - b(Y_t))\,dt + 2e_t\,dW_t$$

with a one-dimensional Brownian motion W_t.

Standard argument: $\|\mu p_t - \nu p_t\|_{TV} \leq \mathbb{P}[T > t]$

UPPER BOUND FOR COUPLING DISTANCE $r_t := |X_t - Y_t|$

$$\begin{aligned} dr_t &= e_t \cdot (b(X_t) - b(Y_t))\,dt \;+\; 2\,dW_t \\ &= \frac{(X_t - Y_t)\cdot(b(X_t) - b(Y_t))}{|X_t - Y_t|^2}\, r_t\,dt \;+\; 2\,dW_t \\ &\le -\frac{1}{2}\,\kappa(r_t)\,r_t\,dt \;+\; 2\,dW_t \qquad \text{for } t < T, \text{ where} \end{aligned}$$

$$\kappa(r) \;:=\; \inf_{|x-y|=r} \frac{2(x-y)\cdot(b(y)-b(x))}{|x-y|^2}$$

EXAMPLE (Langevin diffusion). If $b(x) = -\frac{1}{2}\nabla U(x)$, $U \in C^2(\mathbb{R})$, then

$$\kappa(r) \;=\; \inf_{|x-y|=r} \frac{(x-y)\cdot(\nabla U(x) - \nabla U(y))}{|x-y|^2} \;=\; \inf_{|x-y|=r} \int_0^1 \partial^2_{\frac{x-y}{|x-y|}} U((1-t)x+ty)\,dt.$$

EXAMPLE (Langevin diffusion).

$$dr_t \;\leq\; -\frac{1}{2}\,\kappa(r_t)\,r_t\,dt \;+\; 2\,dW_t \qquad \text{for } t < T\text{, where}$$

$$\kappa(r) \;=\; \inf_{|x-y|=r} \int_0^1 \partial^2_{\frac{x-y}{|x-y|}} U((1-t)x+ty)\,dt.$$

In particular,

- *Convex case*: $\kappa(r) \geq K > 0$
  $\Rightarrow$ Exponential contractivity with rate $K/2$

- *Brownian motion*: $b \equiv 0 \;\Rightarrow\; \kappa \equiv 0$
  $\Rightarrow\; r_t$ local martingale up to T $\Rightarrow$ NO CONTRACTIVITY !

2 Contractivity w.r.t. concave distance functions

Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and concave with $f(0) = 0$.

$$\begin{aligned} df(r_t) &= f'(r_t)\, dr_t + \frac{1}{2} f''(r_t)\, d[r]_t \\ &\leq \text{local martingale} - \frac{1}{2}\kappa(r_t)\, r_t\, f'(r_t)\, dt + 2f''(r_t)\, dt\,. \end{aligned}$$

- Contractivity may hold even if $\kappa < 0$ locally !
- How to choose f for optimal contraction properties ?

ASSUMPTION : $\liminf_{r\to\infty} \kappa(r) > 0.$

$$\begin{aligned} R_0 &:= \inf\{R \geq 0 \,:\, \kappa(r) \geq 0 \;\forall\, r \geq R\}, \\ R_1 &:= \inf\{R \geq R_0 \,:\, \kappa(r) R(R - R_0) \geq 8 \;\forall\, r \geq R\}. \end{aligned}$$

How to choose f for optimal contraction properties ?

$$df(r_t) \;\le\; \text{local martingale} \quad + \underbrace{\left(2f''(r_t) \;-\; \frac{1}{2}\kappa(r_t)\, r_t\, f'(r_t)\right)}_{\overset{!}{\le}\, 0 \;\text{ resp. }\; \overset{!}{\le}\, -c\cdot f(r_t)} dt\,.$$

STEP 1: CONTRACTIVITY

$$f'(r) \;=\; \exp\left(-\frac{1}{4}\int_0^r s\kappa(s)^-\, ds\right) \;=:\; \varphi(r)$$

$\Rightarrow$ $f(r_t)$ supermartingale $\Rightarrow$ $\mathcal{W}_f(\mu p_t, \nu p_t)$ decreasing.

EXAMPLE. If $\kappa^-(r) \equiv L$ for $r \le R$ then

$$\varphi(r) \;=\; \exp(-Lr^2/8) \qquad \text{for } r \le R.$$

STEP 2: EXPONENTIAL CONTRACTIVITY WITH RATE c

$$2f''(r) - \frac{1}{2}\kappa(r)\, r\, f'(r) \overset{!}{\leq} -c\, f(r)$$

ANSATZ:

$$f'(r) = \varphi(r) \cdot g(r), \qquad \varphi(r) = \exp\left(-\frac{1}{4}\int_0^r s\kappa(s)^- \, ds\right),$$

with $g : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing with $g(0) = 1$ and $g(r) = 1/2$ for $r \geq R_1$.

($\Rightarrow$ f concave with $f'(0) = 1$ and $f'(r) \geq \varphi(r)/2 \geq \varphi(R_1)/2$)

REMARK. (i) Here we loose at most a factor 2 ! Indeed, $g(r) \in [0, 1]$ is required anyway in order to guarantee that f is concave and increasing. We are instead assuming $g(r) \in [1/2, 1]$.

(ii) $\varphi/2 \leq f' \leq \varphi \;\Rightarrow\; \Phi/2 \leq f \leq \Phi \;\Rightarrow\; \mathcal{W}_\Phi/2 \leq \mathcal{W}_f \leq \mathcal{W}_\Phi$, where

$$\Phi(r) := \int_0^r \varphi(t)\, dt = \int_0^r \exp\left(-\frac{1}{4}\int_0^t s\kappa(s)^- \, ds\right) dt.$$

STEP 2: EXPONENTIAL CONTRACTIVITY WITH RATE c

$$2f''(r) - \frac{1}{2}\kappa(r)\, r\, f'(r) \overset{!}{\leq} -c\, f(r)$$

ANSATZ: $f'(r) = \varphi(r) \cdot g(r) \quad \Rightarrow$

$$f'' - \frac{1}{4}\kappa r f' = (\varphi g)' - \frac{1}{4}\kappa^- r \varphi g - \frac{1}{4}\kappa r \varphi g \leq \varphi g' \overset{!}{\leq} -\frac{1}{2} c f$$

This holds true if $g' \leq -\frac{1}{2} c\, \Phi/\varphi$. Borderline case:

$$g(r) = 1 - \frac{c}{2}\int_0^r \frac{\Phi(t)}{\varphi(t)}\, dt$$

- Exponential contractivity with rate c holds provided we can choose $f(r) = \int_0^r g(t)\varphi(t)\, dt$ with φ and g as defined above.
- This is possible with $g(r) \geq 1/2$ for $r \leq R_1$ if

$$c \leq 1 \Bigg/ \int_0^{R_1} \Phi(t)\, \varphi(t)^{-1}\, dt.$$

THEOREM. Let $c := 1 \Big/ \int_0^{R_1} \Phi(t)\, \varphi(t)^{-1}\, dt$ and $f(r) := \int_0^r g(t)\varphi(t)\, dt$ with φ and $g = g_c$ as defined above. Then

$$\begin{aligned} \mathcal{W}_f(\mu p_t,\, \nu p_t) &\leq e^{-ct}\, \mathcal{W}_f(\mu, \nu), \qquad \text{and} & (1) \\ W(\mu p_t,\, \nu p_t) &\leq 2\, \varphi(R_0)^{-1}\, e^{-ct}\, \mathcal{W}(\mu, \nu) & (2) \end{aligned}$$

for any $t \geq 0$, and for any probability measures μ, ν on $\mathcal{B}(\mathbb{R}^d)$.

3 Examples

1. *Strictly convex case:* $\kappa(r) \geq K > 0 \ \forall\, r > 0$.

$$c = K/4, \qquad \text{(optimal up to factor } 2 \text{)}$$

2. *Convex case:* $\kappa(r) \geq 0 \ \forall\, r > 0$.

$$c = \min(R^{-2},\, K/4), \qquad \text{(optimal up to factor } 2\pi^2 \text{)}$$

3. *Locally non-convex case:* $\kappa(r) \geq -L \ \forall\, r \leq R$, $\kappa(r) \geq K > 0 \ \forall\, r > R$.

$$c \sim \min(R^{-2},\, K) \qquad \text{if} \quad LR^2 \leq 8,$$
$$c \sim RL^{3/2} \exp\left(-LR^2/8\right) \qquad \text{if} \quad LR^2 \geq 8,\ K > L$$

(again optimal up to constant factor)

4 References

- *T. Lindvall, L.C.G. Rogers 1986*: Coupling of multidimensional diffusions by reflection, Ann. Prob. 14.
- *M.-F. Chen, F.-Y. Wang 1997*: Estimation of spectral gap for elliptic operators, Trans. AMS 349.
- *A.E. 2011*: Reflection coupling and Wasserstein contractivity without convexity, to appear in C.R.Acad.Sci. Paris.