Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 12.

Hand in solutions before Wednesday 9.7., 2 pm.

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1. (Transformations of Brownian motion)

a) Show that the projection of a d-dimensional Brownian motion onto a hyperplane yields a one-dimensional Brownian motion: Suppose $(W_t^{(1)}, \dots, W_t^{(d)})$ is a d-dimensional Brownian motion started from 0 and $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ with $\sum_{i=1}^d \lambda_i^2 = 1$. Show that $X_t = \sum_{i=1}^d \lambda_i W_t^{(i)}$ is a Brownian motion started from 0.

b) Show that rotating a Brownian motion about the origin yields another Brownian motion: Let W be a d-dimensional Brownian motion started at 0 and let A be a $d \times d$ orthogonal matrix. Show that $Y_t = AW_t$ is again a d-dimensional Brownian motion.

c) Formulate a statement that includes a) and b) !

2. (Stopped Brownian motion) Let X_t be a one-dimensional Brownian motion started at 0 and let $T = \min\{t : |X_t| = 1\}$ and $T^* = \min\{t : X_t = 1 \text{ or } X_t = -3\}$.

a) Explain why X_T and T are independent random variables.

b) Show that T^* and X_{T^*} are not independent.

3. (Local maxima of Brownian paths) Let B_t be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$. Show that the following statements hold for almost every ω:

a) The trajectory $t \mapsto B_t(\omega)$ is not monotone in any interval $[a, b]$ with $a < b$.

b) The set of local maxima of $t \mapsto B_t(\omega)$ is dense in $[0, \infty)$.

c) All local maxima of $t \mapsto B_t(\omega)$ are strict (i.e., for any local maximum m there exists an $\varepsilon > 0$ such that $B_t(\omega) < B_m(\omega)$ for all $t \in (m - \varepsilon, m + \varepsilon)$).

4. (Wiener–Lévy representation and quadratic variation)
The quadratic variation $[x]_t$ of a continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ along the sequence of dyadic partitions of the intervals $[0, t]$ is defined by

$$[x]_t := \lim_{m \to \infty} \sum_{i=1}^{2^m} \left| x(t_i^{(m)}) - x(t_{i-1}^{(m)}) \right|^2; \qquad t_i^{(m)} = i 2^{-m} t.$$

a) Show that the quadratic variation of a continuously differentiable function x vanishes, i.e., $[x]_t = 0$ for any $t \geq 0$.

b) Let

$$x(t) = x(1) \cdot t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k} \cdot e_{n,k}(t), \qquad a_{n,k} \in \mathbb{R},$$

be the expansion of a function $x \in C([0,1])$ with $x(0) = 0$ in the basis of Schauder functions. Show that

$$[x]_1 = \lim_{m \to \infty} \frac{1}{2^m} \sum_{n=0}^{m-1} \sum_{k=0}^{2^n-1} (a_{n,k})^2.$$

c) Deduce that almost every path of Brownian motion has quadratic variation $[B]_t = t$. Why does it suffice to consider $t = 1$?

d) Determine the quadratic variation of the "self-similar" function

$$g(t) := t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} e_{n,k}(t)$$

on the interval $[0,1]$, and on $[0,t]$ for $t \in [0,1)$. Compare with the results from b) and c).