Institute for Applied Mathematics
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"Stochastic Processes", Problem Sheet 11.

Hand in solutions before Wednesday 2.7., 2 pm.

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1. (Probabilities of Brownian motion)

Let B_t $(t \geq 0)$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ with $B_0 = 0$.

a) Compute the probabilities of the following events:

(i) $B_2 > 2$.

(ii) $B_2 > B_1$.

(iii) $B_2 > B_1 > B_3$.

b) Let $Z := \sup_{t \geq 0} B_t$. Show that λZ has the same distribution as Z for any $\lambda > 0$. Deduce that $Z = +\infty$ P–a.s.

2. (Symmetries of Brownian motion)

Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be independent one-dimensional Brownian motions starting at 0.

a) Show that the following processes are Brownian motions:

$$\text{(i)} \ -X_t \qquad \text{(ii)} \ X_{t+h} - X_h \ \ (h \geq 0 \text{ fixed}) \qquad \text{(iii)} \ \frac{1}{\sqrt{a}} X_{at} \ \ (a > 0 \text{ fixed}).$$

b) Show that $B_t := (X_t - Y_t)/\sqrt{2}$ is a Brownian motion.

c) True or false: With probability 1 we have $X_t = Y_t$ for infinitely many $t > 0$.

3. (Wiener-Lévy Representation of Brownian Motion)

The Schauder functions $e_{n,k} \in C([0,1])$ are defined in the following way :

$$e_{0,1}(t) := \min(t, 1-t),$$
$$e_{n,k}(t) := \begin{cases} 2^{-n/2} \cdot e_{0,1}(2^n t - k) & \text{for } t \in [k2^{-n}, (k+1)2^{-n}], \\ 0 & \text{otherwise,} \end{cases}$$

$n \in \mathbb{N}$, $k = 0, 1, 2, \ldots, 2^n - 1$. For $x \in C([0,1])$ with $x(0) = 0$ let

$$a_{n,k} := 2^{n/2} \cdot \Delta_{n,k} x \qquad \text{with} \qquad \Delta_{n,k} x := 2 \cdot (x(m_{n,k}) - \bar{x}_{n,k}),$$

where $m_{n,k}$ denotes the midpoint of the dyadic interval $[k2^{-n}, (k+1)2^{-n}]$, and $\bar{x}_{n,k} := (x((k+1) \cdot 2^{-n}) + x(k \cdot 2^{-n}))/2$. Show that :

a) The sequence

$$x_m(t) \;:=\; x(1)\cdot t \;+\; \sum_{n=0}^{m}\sum_{k=0}^{2^n-1} a_{n,k}\cdot e_{n,k}(t), \qquad m\in\mathbb{N},$$

converges to $x(t)$ uniformly for $t\in[0,1]$. *(Hint : Verify that x_m is the polygonal approximation of x w.r.t. the m–th dyadic partition of the interval $[0,1]$)*

b) Under Wiener measure μ_0 on $\Omega = C([0,1])$, the random variables

$$X_1(\omega) \qquad \text{und} \qquad Y_{n,k}(\omega) \;:=\; 2^{n/2}\cdot\Delta_{n,k}X(\omega) \qquad (n\geq 0, 0\leq k<2^n),$$

are independent with distribution $N(0,1)$, and the *Wiener–Lévy Representation*

$$X_t(\omega) \;=\; X_1(\omega)\cdot t \;+\; \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} Y_{n,k}(\omega)\cdot e_{n,k}(t) \qquad \text{holds for any } \omega\in\Omega.$$

c) How can this be used in order to simulate sample paths of Brownian motion ?

4. (World Cup 2014)

Model the sequence of goals during the first 90 minutes of a world cup match of the German national team by a continuous time Markov chain. Estimate the transition rates using the data of the last 25 world cup matches. Discuss strengths and weaknesses of your model, and simulate the sequence of goals in the next match.

Germany - Mexico 2:1. Goals: 0:1(47.), 1:1(75.), 2:1(86.).
Germany - Croatia 0:3. Goals: 0:1(45.), 0:2(80.), 0:3(85.).
Germany - Saudi Arabia 8:0. Goals: 21., 26., 40., 45., 70., 72., 85., 90.
Germany - Ireland 1:1. Goals: 1:0(19.), 1:1(90.).
Germany - Cameroons 2:0. Goals: 1:0(50.), 2:0(79.).
Germany - Paraguay 1:0. Goals: 1:0(88.).
Germany - USA 1:0. Goals: 1:0(39.).
Germany - South Korea 1:0. Goals: 1:0(75.).
Germany - Brasil 0:2. Goals: 0:1(67.), 0:2(79.).
Germany - Costa Rica 4:2. Goals: 1:0(6.), 1:1(12.), 2:1(17.), 3:1(61.), 3:2(73.), 4:2(87.).
Germany - Poland 1:0. Goals: 1:0(90.).
Germany - Ecuador 3:0. Goals: 1:0(4.), 2:0(44.), 3:0(57.).
Germany - Sweden 2:0. Goals: 1:0(4.), 2:0(12.).
Germany - Argentina 1:1. Goals: 0:1(49.), 1:1(80.).
Germany - Italy 0:0.
Germany - Portugal 3:1. Goals: 1:0(56.), 2:0(61.), 3:0(78.), 3:1(88.).
Germany - Australia 4:0. Goals: 1:0(8.), 2:0(26.), 3:0(68.), 4:0(70.).
Germany - Serbia 0:1. Goals: 0:1(38.).
Germany - Ghana 1:0. Goals: 1:0(60.).
Germany - England 4:1. Goals: 1:0(20.), 2:0(32.), 2:1(38.), 3:1(67.), 4:1(70.).
Germany - Argentina 4:0. Goals: 1:0(3.), 2:0(68.), 3:0(74.), 4:0(89.).
Germany - Spain 0:1. Goals: 0:1(73.).
Germany - Portugal 4:0. Goals: 1:0(12.), 2:0(32.), 3:0 (45.), 4:0 (78.).
Germany - Ghana 2:2. Goals: 1:0(61.), 1:1(54.), 1:2 (63.), 2:2 (71.).
Germany - USA .... . Goals: ....