Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 10.

Hand in solutions before Wednesday 25.6., 2 pm.

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1. (Hitting times for the 2-dimensional random walk) Let Z_n be the random walk on $\mathbb{Z}^2$ starting in z_0 and making a step in one of the four directions with equal probability.

a) Show that $|Z_n|^2 - n$ is a martingale.

b) For $r > |z_0|$ let
$$T \;:=\; \inf\{n \geq 0 : |Z_n|^2 \geq r^2\}$$
be the exit time from the circle around 0 with radius r. Prove that
$$r^2 - |z_0|^2 \;\leq\; E[T] \;\leq\; (r+1)^2 - |z_0|^2 \,.$$

2. (Martingales of Brownian motion) State the definition of a martingale in continuous time. Show that the following processes are martingales:

a) A one-dimensional Brownian motion $(B_t)_{t\geq 0}$ w.r.t. $\mathcal{F}_t = \sigma(B_r \,:\, 0 \leq r \leq t)$.

b) $M_t^\lambda \;=\; \exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$, $\lambda \in \mathbb{R}$, w.r.t. the same filtration.

c) $h(B_t)$, if $(B_t)_{t\geq 0}$ is a d–dimensional Brownian motion, and h is a harmonic function on $\mathbb{R}^d$, w.r.t. $\mathcal{F}_t = \sigma(B_r^1, B_r^2, \ldots, B_r^d \,:\, 0 \leq r \leq t)$.

3. (Zeros of Brownian paths)
Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose sample paths $t \mapsto B_t(\omega)$ are all continuous.

a) Show that $(t, \omega) \mapsto B_t(\omega)$ is measureable as a map from $\Omega \times [0, 1]$ to $\mathbb{R}$.

b) Compute the expectation and the variance of $\int_0^1 B_s(\omega)\, ds$.

c) Show that $\mathbb{P}$–a.s.: $\lambda\left[\{t \in [0, 1] |\, B_t(\omega) = 0\}\right] = 0\,.$

4. (Random signs) Let (a_n) be a sequence of real numbers with $\sum a_n^2 = \infty$, and let
$$M_n \;=\; \sum_{k=1}^n \varepsilon_k a_k\,, \qquad \varepsilon_k \text{ i.i.d. with } \quad P[\varepsilon_k = \pm 1] = 1/2.$$

a) Determine the conditional variance process $\langle M \rangle_n$.

b) For $c > 0$ let $T_c := \inf\{n \geq 0 : |M_n| \geq c\}$. Show that $P[T_c < \infty] = 1$.

c) Conclude that almost surely, the process (M_n) has unbounded oscillations.