Institute for Applied Mathematics
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"Stochastic Processes", Problem Sheet 9.

Hand in solutions before Wednesday 18.6., 2 pm.

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1. (Ruin problem for the asymmetric random walk revisited)
Let $p \in (0,1)$ with $p \neq 1/2$. We consider the random walk $S_n = Y_1 + \cdots + Y_n$, Y_i $(i \geq 1)$ i.i.d. with $P[Y_i = +1] = p$ and $P[Y_i = -1] = q := 1 - p$.

a) Show that the following processes are martingales:

$$M_n \;:=\; (q/p)^{S_n} \;, \qquad N_n \;:=\; S_n - n(p-q)\,.$$

b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ let $T := \min\{n \geq 0 \,|\, S_n \notin (a,b)\}$. Deduce from a) that

$$P[S_T = a] \;=\; \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,, \quad \text{and} \quad E[T] \;=\; \frac{b}{p-q} - \frac{b-a}{p-q} \cdot \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,.$$

2. (Star Trek I) The control system on the star-ship *Enterprise* has gone wonky. All that one can do is to set a distance to be travelled. The spaceship will then move that distance in a randomly chosen direction, then stop. The object is to get into the Solar System, a ball of radius r. Initially, the *Enterprise* is at a distance $R_0 (> r)$ from the Sun. Let R_n be the distance from Sun to *Enterprise* after n 'space-hops'.

a) Show that, for any strategy which always sets a distance not greater than that from the Sun to the *Enterprise*, $1/R_n$ is a martingale. *(Hint: Use the mean-value property of harmonic functions (Proof?): If $\Delta f = 0$ on a ball $B \subset \mathbb{R}^3$, then $f(x) = \oint_{\partial B} f$)*

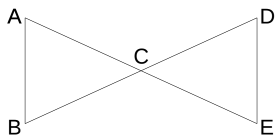
b) Deduce that $P[$*Enterprise* gets into Solar System$] \leq r/R_0$.

c) For each $\varepsilon > 0$, you can choose a strategy which makes this probability greater than $r/R_0 - \varepsilon$. What kind of strategy will that be?

3. (Random walk on a bow tie) A particle performs a random walk on a bow tie ABCDE drawn beneath. From any vertex its next step is equally likely to be to any neighbouring vertex. Initially it is at A. Find the expected value of:

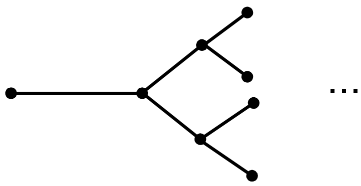
a) the time of first return to A, given no prior visit by the particle to E,

b) the number of visits to D before returning to A, given no prior visit to E.



4. (Recurrence and transience of random walks)

a) Is the simple random walk on the infinite binary tree



recurrent or transient ?

b) Prove that: For the classical random walk on $\mathbb{Z}^3$ we have

$$p^{2n}(x,x) \;=\; \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i\,j\,k}^2\left(\frac{1}{3}\right)^{2n} \;\leq\; \frac{const.}{n^{3/2}},$$

where $\binom{n}{i\,j\,k} := \frac{n!}{i!j!k!}$. Deduce that the random walk is transient.

Hint : $\sum_{i+j+k=n}\binom{n}{i\,j\,k} = 3^n$.

5. (CRR model of stock market) In the Cox-Ross-Rubinstein binomial model of mathematical finance, the price of an asset is changing during each period either by a factor $1+a$ or by a factor $1+b$ with $a,b\in(-1,\infty)$ such that $a<b$. We can model the price evolution in N periods by a stochastic process

$$S_n = S_0\cdot\prod_{i=1}^{n} X_i, \qquad n=0,1,2,\dots,N,$$

defined on $\Omega=\{1+a,1+b\}^N$, where the initial price S_0 is a given constant, and $X_i(\omega)=\omega_i$. Taking into account a constant interest rate $r>0$, the discounted stock price after n periods is

$$\tilde{S}_n \;=\; S_n/(1+r)^n.$$

A probability measure P on Ω is called a *martingale measure* if the discounted stock price is a martingale w.r.t. P and the filtration $\mathcal{F}_n=\sigma(X_1,\dots,X_n)$. Martingale measures are important for option pricing under no arbitrage assumptions.

a) Show that P is a martingale measure if and only if the growth factors $X_1,\dots,X_N$ are independent with

$$P[X_n=1+b] \;=\; \frac{r-a}{b-a} \qquad\text{and}\qquad P[X_n=1+a] \;=\; \frac{b-r}{b-a}.$$

Conclude that a (unique) martingale measure exists if and only if $r\in[a,b]$.

b) Show that in a corresponding trinomial model, i.e. $\Omega=\{1+a,1+b,1+c\}^N$, there may be infinitely many martingale measures.