Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 8.

Hand in solutions before Wednesday 4.6., 2 pm.

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1. (Ruin probabilities) A player has 2 Euros, and he wants to make 10 Euros out of it as fast as possible. He plays a game with the following rules: In each round a fair coin is tossed. If the coin comes up heads, the player will win a sum as high as his stake, and he will get his stake back. Otherwise he will lose his stake. The player decides to choose a fixed strategy: If he owns less than 5 Euros, he bets his whole capital; otherwise he bets exactly the amount needed to reach 10 Euros in case he wins. Show that the player achieves his goal with probability 1/5.

2. (Dichotomy of transience and recurrence) Consider a time homogeneous Markov chain (X_n, P_x) with transition kernel p on a countable set S. For $y \in S$ let

$$B_y(\omega) = \sum_{n=0}^{\infty} 1_{\{y\}}(X_n(\omega))$$

denote the number of visits of the Markov chain to the state y, and let

$$T_y = \min\{n \geq 1 : X_n = y\}$$

denote the first passage/return time to the state y.

a) Prove that $P_x[B_x \geq n] = P_x[T_x < \infty]^n$.

b) Prove the following dichotomy: Either

$$P_x[T_x < \infty] = 1 \quad \text{and} \quad B_x = \infty \ P_x\text{-a.s.} \quad \text{and} \quad G(x,x) = \infty \quad \textbf{(recurrence)},$$

or

$$P_x[T_x < \infty] < 1 \quad \text{and} \quad B_x < \infty \ P_x\text{-a.s.} \quad \text{and} \quad G(x,x) < \infty \quad \textbf{(transience)}.$$

c) Consider a random walk on $\mathbb{Z}^d$ given by

$$X_{n+1} = X_n + U_{n+1},$$

where the random varaibles $U_i,\ i = 1, 2, \ldots,$ are independent and uniformly distributed on $\{-1, +1\}^d$. Study recurrence and transience of the random walk depending on the dimension d.

3. (Martingales) A process $(X_n)_{n\in\mathbb{Z}_+}$ is called *predictable* w.r.t. a filtration $(\mathcal{F}_n)$ iff X_n is measurable w.r.t. $\mathcal{F}_{n-1}$ for any $n \in \mathbb{N}$. Show that:

a) A predictable martingale is almost surely constant.

b) For a nonnegative martingale $(X_n)_{n\geq 0}$ we have almost surely:

$$X_n(\omega) = 0 \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \text{ for any } k \geq 0 \ .$$

4. (Optional stopping) Let $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration.

a) Prove by induction: If (X_n) is a martingale and T is a stopping time w.r.t. $(\mathcal{F}_n)$ then

$$E[X_{T\wedge n}] \;=\; E[X_0] \qquad \forall\, n \geq 0 \ .$$

b) Give sufficient conditions such that

$$E[X_T] \;=\; E[X_0].$$