Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 7.

Hand in solutions before Wednesday 28.5., 2 pm.

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1. (Distribution of the first return time)

Let (X_n, P_x) be a Markov chain on a countable state space S, and let $T_x := \min\{n \geq 1 : X_n = x\}$. The generating function of the distribution of T_x when starting in x is

$$G(z) \;=\; E_x\left[z^{T_x}\right] \qquad (\, z \in \mathbb{C} \text{ with } |z| < 1\,)\,.$$

a) Show that:

$$\sum_{n=0}^{\infty} P_x[X_n = x]\, z^n \;=\; \sum_{k=0}^{\infty} E_x\left[z^{T^{(k)}}\right] \;=\; \frac{1}{1 - G(z)}$$

where $T^{(k)}$ is the k–th return time to x.

b) Deduce that for the simple random walk on $\mathbb{Z}$ we have

$$\frac{1}{1 - G(z)} \;=\; \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} z^{2n} \;=\; \frac{1}{\sqrt{1 - z^2}}\,,$$

hence $G(z) = 1 - \sqrt{1 - z^2}$. In particular $E_x[T_x] = \infty$.

2. (Random walks on discrete circles)

Let $k \in \mathbb{N}$ with $p, q, r \geq 0$ and $p + q + r = 1$. Consider the random walk on $\mathbb{Z}/k\mathbb{Z}$ with transition probabilities

$$p(x, x+1) = p,\;\; p(x, x) = r,\;\; p(x, x-1) = q,\;\; \text{and } p(x, y) = 0 \text{ otherwise.}$$

Depending on the parameters p, q, r, k:

a) Determine all stationary distributions.

b) Study the convergence to a stationary distribution.

3. (Contractivity in total variation distance)

a) Let p be a stochastic kernel on a measurable space $(S, \mathcal{S})$. Show that for any two probability measures μ, ν on S, we have

$$d_{TV}(\mu p, \nu p) \leq d_{TV}(\mu, \nu).$$

b) The "House of cards" is the Markov chain with state space $\mathbb{Z}_+$ and transition probabilities $p(x, x+1) = 1 - \varepsilon$, $p(x, 0) = \varepsilon$, where $\varepsilon \in (0, 1)$ is a fixed constant. Show that

$$\exists\, \alpha \in (0,1) : \ d_{TV}(\mu p, \nu p) \leq \alpha\, d_{TV}(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{Z}_+). \tag{1}$$

Hence conclude that there is a unique stationary distribution $\bar{\mu}$ of p such that $\mathrm{Law}(X_n) \to \bar{\mu}$ in total variation for any initial distribution μ.

c) Does (1) also hold for the AR(1) process on $\mathbb{R}^1$?

4. (Extinction probabilities for Birth-and-death chains)

Let (X_n, P_x) be the canonical Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities

$$p(x, x+1) = p_x, \ p(x, x) = r_x, \ p(x, x-1) = q_x, \ \text{and } p(x, y) = 0 \text{ otherwise},$$

where $p_x + q_x + r_x = 1, \ q_0 = 0,$ and $p_x, q_x \neq 0 \ \forall\, x \neq 0$.

a) Deduce from the mean-value property

$$p_x u(x+1) + r_x u(x) + q_x u(x-1) \ = \ u(x) \qquad \forall\, x \geq 1$$

an equivalent equation for the differences $v(x) := u(x+1) - u(x)$. Hence determine all harmonic functions for the Markov chain.

b) Show that for $0 \leq a < b$,

$$P_x\left[X_{T_{a,b}} = a\right] \ = \ \frac{h(b) - h(x)}{h(b) - h(a)} \qquad \forall\, a \leq x \leq b\,,$$

where $T_{a,b} = \inf\{n \geq 0 \ : \ X_n \notin (a, b)\,\}$, and

$$h(x) \ = \ \sum_{y=0}^{x-1} \prod_{z=1}^{y} \frac{q_z}{p_z}.$$

c) Compute the extinction probability $P_x[\exists\, n \geq 0 : \ X_n = 0]$ when starting in x. Under which condition does the process become extinct almost surely? What happens asymptotically in the other cases?