Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 6.

Hand in solutions before Wednesday 21.5., 2 pm.

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1. (First step analysis)

We consider the Random Walk on $\mathbb{Z}$ with transition probabilities $p(x, x+1) = p$ and $p(x, x-1) = q := 1 - p$ where $p \in (\frac{1}{2}, 1)$. Let

$$u(x) \; := \; E_x \left[\sum_{n=0}^{\infty} a^{X_n} \right] \quad , \qquad a > 0 \, .$$

a) Show that $u(x+1) \; = \; a \cdot u(x) \, .$

b) Compute $u(0)$ by conditioning on the first step, and interpret the result.

2. (Finite Markov chains)

We consider a time-homogeneous Markov chain on $\{1, 2, 3\}$ starting in the state x with transition matrix

$$p := \begin{pmatrix} 1-2q & 2q & 0 \\ q & 1-2q & q \\ 0 & 2q & 1-2q \end{pmatrix} .$$

For $x = 1, 2, 3$ compute

a) the n–step return probabilities $P[X_n = x]$,

b) the average number of returns to the starting point x until time n.

What is the relative frequency of visits to the starting point in the limit as $n \to \infty$?

3. (Stationary distributions of autoregressive processes in $\mathbb{R}^d$)

An AR(1) process (X_n) in $\mathbb{R}^d$ is given by a recursion

$$X_n \; = \; \theta X_{n-1} + U_n,$$

where $\theta : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear map, and the random variables U_n are independent and $N(0, I_d)$ distributed.

Prove that: If θ is not symmetric, then under suitable assumptions to be specified, there exists a stationary distribution, but the "Detailed Balance" condition is not satisfied!

4. (Stationary distributions in proof-reading)

The proof copy of a book is read by an infinite sequence of editors checking for mistakes. Each mistake is detected with probability p at each reading; between readings the printer corrects the detected mistakes, but introduces a random number of new errors (errors may be introduced even if no mistakes were detected). We assume that the number of new errors introduced after different readings are Poisson distributed with parameter λ.

a) Let G_n denote the generating function of the number X_n of errors after the nth editor-printer cycle. Compute G_{n+1} in terms of G_n assuming as much independence as necessary. Conclude that

$$G_n(u) \;=\; \exp\left(\lambda(u-1)\sum_{k=0}^{n-1} p^k\right) G_0(1-p^n+p^n u).$$

b) Show that $\rho(u) = \lim_{n\to\infty} G_n(u)$ exists and does not depend on G_0, and that ρ is the generating function of a Poisson distribution. Hence find the stationary distribution(s) of the Markov chain (X_n).