Institute for Applied Mathematics
Summer term 2014
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“Stochastic Processes”, Problem Sheet 5.

Hand in solutions before Wednesday 14.5., 2 pm.

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1. (Examples of Markov chains) A dice is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

a) The largest number M_n shown up to the nth roll.

b) The number N_n of sixes in n rolls.

c) At time r, the time C_r since the most recent six.

d) At time r, the time B_r until the next six.

2. (Stochastic processes constructed from Bernoulli random variables) Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with

$$\mathbb{P}[\beta_n = 1] = p, \quad \mathbb{P}[\beta_n = 0] = q, \text{ where } p + q = 1.$$

a) We define the stochastic process (X_n) for $n = 2, 3, \ldots$ by

$$X_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{if } \beta_{n-1} = 1, \beta_n = 0, \\ 2 & \text{if } \beta_{n-1} = 0, \beta_n = 1, \\ 3 & \text{if } \beta_{n-1} = \beta_n = 0. \end{cases}$$

(i) Prove that the process $(X_n)_{n\geq 2}$ is a Markov chain.

(ii) Compute the transition matrix P.

(iii) Compute the probability $\mathbb{P}[X_{n+3} = 0 \,|\, X_n = 0]$.

b) We define the stochastic process (Y_n) for $n = 2, 3, \ldots$ by

$$Y_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Show that the process $(Y_n)_{n\geq 2}$ is not a Markov chain.

3. (A Markov chain on $\{1, 2, 3\}$) We consider the Markov chain $(X_n)_{n=0,1,2,\cdots}$ with state space $S = \{1, 2, 3\}$, initial state $X_0 = 2$, and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & 1-p-q & q \\ 0 & 0 & 1 \end{pmatrix}, \qquad p, q > 0, \; p + q < 1.$$

a) Show that (X_n) first changes its value at a random time $T \geq 1$ whose law is geometric.

b) Show also that X_T is independent of T, and give the law of X_T.

c) Finally show that $X_t = X_T$ almost surely for $t \geq T$.

4. (Superposition and thinning of Poisson processes)
Let ν and $\tilde{\nu}$ be finite measures on a measurable space $(S, \mathcal{S})$, and let $\alpha : S \to [0,1]$ be a measurable function.

a) Show that if N and $\tilde{N}$ are independent spatial Poisson processes with intensity measures ν, $\tilde{\nu}$ respectively, then $N + \tilde{N}$ is a Poisson process with intensity $\nu + \tilde{\nu}$.

b) Let $Z, X_1, X_2, \ldots$, and $U_1, U_2, \ldots$ be independent random variables with distributions $Z \sim \text{Poisson}(\nu(S))$, $X_i \sim \nu/\nu(S)$, and $U_i \sim \text{Unif}(0,1)$. Show that

$$N_\alpha = \sum_{i=1}^{Z} \mathbf{1}_{\{U_i \leq \alpha(X_i)\}} \, \delta_{X_i}$$

is a Poisson process with intensity measure $\alpha(x)\, \nu(dx)$.

5. (*Age-dependent branching process)
Consider a population model where each individual lives for a period of time (called "age") before it gives birth to its family of next-generation descendants. We assume that the family sizes and the ages are all independent, that the family sizes are identically distributed on $\mathbb{Z}_+$ with generating function G, and the ages are identically distributed on $(0, \infty)$ with density function f. Let X_t denote the size of the population at time t where $X_0 := 1$. The generating function of the population size is now also a function of t:

$$G_t(s) \;:=\; \mathbb{E}\left[s^{X_t}\right], \qquad s \in [0,1],\; t \in [0, \infty).$$

a) Explain intuitively that

$$X_{T+t} \;\sim\; \sum_{i=1}^{Z} X_t^{(i)} \qquad \text{for any } t \geq 0,$$

where $T(\omega)$ is the first branching time, the processes $(X_t^{(i)})$, $i \in \mathbb{N}$, are independent copies of (X_t), and Z is independent of the $X^{(i)}$ with generating function G.

b) Conclude that for $t \geq 0$ and $s \in [0,1]$,

$$G_t(s) \;=\; \int_0^t G(G_{t-u}(s))\, f(u)\, du + \int_t^\infty s\, f(u)\, du.$$

c) Now assume that the ages are exponentially distributed with parameter λ. Show that

$$\frac{\partial}{\partial t} G_t(s) \;=\; \lambda \left[G(G_t(s)) - G_t(s)\right].$$

Conclude that in the case of binary branching, $G(s) = s^2$ and

$$G_t(s) \;=\; \frac{s e^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}.$$