Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 3.

Hand in solutions before Wednesday 30.4., 2 pm.
(post-box opposite to maths library)

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1. (Conditional distributions)

a) The joint density of X and Y is given by $f(x,y) := 1/x,\ 0 \le y \le x \le 1$.

(i) Find regular versions of the conditional distributions of X given Y, and of Y given X.

(ii) Compute $\mathbb{E}[X|Y]$ and $\mathbb{E}[Y|X]$.

b) Let S, T and U be independent exponentially distributed random variables with parameters λ, μ, ν. Show that $\min(T, U)$ is exponentially distributed with parameter $\mu + \nu$, and compute the probabilities $\mathbb{P}[T < U]$ and $\mathbb{P}[S < T < U]$.

2. (Independence and conditional expectations)
Let X, Y be random variables on a joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that X is integrable, and U is independent from the pair (X, Y).

a) Prove that:
$$\mathbb{E}[X|Y,U] \;=\; \mathbb{E}[X|Y] \qquad \mathbb{P}\text{-almost surely.} \tag{1}$$

b) Give an example to show that (1) does not necessarily hold, if one only assumes independence of X and U. Explain this fact intuitively.

3. (Martingales of a simple random walk) Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables with $P[Y_i = \pm 1] = \frac{1}{2}$, and let

$$X_n = x + S_n \quad \text{where } S_n = Y_1 + \cdots + Y_n.$$

Show that the following processes are martingales w.r.t. the filtration given by $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ (see Problem Sheet 2, Exercise 3 for the definition):

a) X_n

b) $M_n = X_n^2 - n$

c) $M_n^\lambda = e^{\lambda X_n - a(\lambda) n}$ for any $\lambda \in \mathbb{R}$, where $a(\lambda) = \log\cosh\lambda$.

4. (Inequalities for conditional expectations)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra.

a) Prove the following generalization of Markov's Inequality:

$$\mathbb{P}\big[|X| \geq \alpha|\mathcal{G}\big] \ \leq \ \frac{1}{\alpha^k}\mathbb{E}\big[|X|^k|\mathcal{G}\big] \quad \mathbb{P}\text{-a.s.}$$

b) State and prove the Cauchy-Schwarz inequality for conditional expectations.