Institute for Applied Mathematics
Summer term 2014
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"Stochastic Processes", Problem Sheet 1.

Hand in solutions before Wednesday 16.4., 2 pm.
(post-box opposite to maths library)

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1. (Conditional Expectations)
Let $X, Y : \Omega \to [0, \infty)$ be independent identically distributed (iid) discrete random variables with expectation m.

a) Find the mistake in the following reasoning:

$$E[X \,|\, X + Y = z] \;=\; E[X \,|\, X = z - Y] \;=\; E[z - Y] \;=\; z - m.$$

b) Show that

$$E[X \,|\, X + Y] \;=\; \frac{1}{2}(X + Y).$$

c) Can one proof in a similar way that

$$E[X \,|\, X \cdot Y] \;=\; (X \cdot Y)^{1/2} \text{ ?}$$

2. (Error Detection) A factory is producing notebooks that are defect with probability p. A test identifies mistakes (if there are any) with probability $1 - \varepsilon$.

a) Show that the probability that a notebook which passed the test is nevertheless defect, is $\varepsilon p/(1 - p + \varepsilon p)$.

b) The factory produces n notebooks a day. Let X denote the number of defect notebooks, and let Y be the number of notebooks identified as defect. Under suitable independence assumptions show that

$$E[X \,|\, Y] \;=\; Y + (n - Y) \cdot \frac{\varepsilon p}{1 - p + \varepsilon p} \;=\; \frac{\varepsilon p n + (1 - p)Y}{1 - p + \varepsilon p}\,.$$

3. (Transformations of exponential random variables)
Let T and R be independent exponentially distributed random variables with parameters λ and μ respectively. Determine

a) the conditional distribution of T given $T + R$,

b) the distribution of T/R.

4. (Properties of conditional expectations)

Let $Y : \Omega \to S$ be a discrete random variable, and let $X : \Omega \to \mathbb{R}$ be an integrable real-valued random variable, both defined on a common probability space $(\Omega, \mathcal{A}, P)$. Prove that:

a) The map $X \to E[X|Y]$ is almost surely linear and monotone.

b) If $X = \tilde{X}$ almost surely, then also $E[X|Y] = E[\tilde{X}|Y]$ almost surely.

c) For any $f : S \to \mathbb{R}$ such that $f(Y) \cdot X \in \mathcal{L}^1$,

$$E[f(Y) \cdot X|Y] \;=\; f(Y) \cdot E[X|Y] \qquad P\text{-a.s.}$$

Which result do we obtain if X and Y are independent ?