1. (Computation of probabilities and expectations [30 points])

Suppose that all the following processes are defined on a common probability space $(\Omega, \mathcal{A}, P)$. Moreover, assume that they are all independent of each other and all processes start at 0: $N_0 = S_0 = B_0 = X_0 = 0$.

Compute the following probabilities and expectations. You can use results from the lecture without proof if you state them carefully.

a) $P[N_t \geq 2]$,
where $(N_t)_{t\geq 0}$ is a Poisson process with intensity 1. *[4]*

b) $E[S_n]$ and $E[S_{N_t}]$,
where $(S_n)_{n\in\mathbb{Z}_+}$ is a Random walk on the integers with transition probabilities $p(x, x+1) = 3/4$, $p(x, x-1) = 1/4$. *[8]*

c) $P[B_5 > B_2]$ and $E\left[\max_{s\in[0,t]} B_s\right]$,
where $(B_t)_{t\geq 0}$ is a one dimensional Brownian motion. *[8]*

d) $E[X_1^2]$, $E[X_2^2]$ and $P[B_2 > X_1]$,
where $(X_n)_{n\in\mathbb{Z}_+}$ is a Markov chain on $\mathbb{R}$ with transition kernel $p(x, \cdot) = N(x, 2)$. *[10]*

Solution. a) We have that $N_t \sim$ Poisson(t). Therefore,

$$P[N_t \geq 2] = 1 - P[N_t = 0] - P[N_t = 1] = 1 - e^{-t}(1 + t).$$

b) We have that $S_n = \sum_{k=1}^n X_k$, where X_k are i.i.d. random variables such that $P[X_k = -1] = 1 - P[X_k = 1] = 1/4$. In particular, $E[X_k] = 1/2$. Then

$$E[S_n] = \sum_{k=1}^n E[X_k] = n/2.$$

Moreover, since (X_k) is independent of N_t, we have that

$$E[S_{N_t}] = E[E[S_{N_t}|N_t]] = \int_\Omega E[S_{N_t}|N_t](\omega)\, P(d\omega) = \int_\Omega E\left[S_{N_t(\omega)}\right]\, P(d\omega) = \frac{E[N_t]}{2} = \frac{t}{2}.$$

c) $P[B_5 > B_2] = P[B_5 - B_2 > 0] = 1/2$, since $B_5 - B_2 \sim N(0, 3)$.
Moreover, by the reflection principle, $P[\max_{s\in[0,t]} B_s \geq a] = 2P[B_t \geq a]$. Hence

$$\begin{aligned}
E[\max_{s\in[0,t]} B_s] &= \int_0^\infty P[\max_{s\in[0,t]} B_s \geq a]\, da = \int_0^\infty 2P[B_t \geq a]\, da \\
&= \int_0^\infty \int_a^\infty 2\varphi_t(x)\, dx\, da = \int_0^\infty \int_0^x 2\varphi_t(x)\, da\, dx = \int_0^\infty 2x\varphi_t(x)\, dx \\
&= 2\int_0^\infty \frac{x}{\sqrt{2\pi t}} \exp(-x^2/2t)\, dx = -\sqrt{\frac{2}{\pi t}}\, t \exp(-x^2/2t)\Big|_0^\infty = \sqrt{\frac{2t}{\pi}}.
\end{aligned}$$

d) Note that $X_1 \sim N(0, 2)$. Therefore, $E[X_1^2] = 2$. Furthermore, since the conditional distribution of X_2 given X_1 is $N(X_1, 2)$, we have $E[X_2] = E[E[X_2|X_1] = E[X_1] = 0$, and

$$E[X_2^2] = \mathrm{Var}[X_2] = E[\mathrm{Var}[X_2|X_1]] + \mathrm{Var}[E[X_2|X_1]] = E[2] + \mathrm{Var}[X_1] = 2 + 2 = 4.$$

Finally, since B_2 and X_1 are independent centred Gaussian random variables, $B_2 - X_1$ is a centred Gaussian random variable as well. Hence,

$$P[B_2 > X_1] = P[B_2 - X_1 > 0] = 1/2.$$

2. (Markov chains [10 points])

Let (X_n, P_x) be a canonical time homogeneous Markov chain with state space $S = \mathbb{Z}_+$ and transition matrix $p = (p(x,y))_{x,y \in S}$.

a) State without proof the Markov property for (X_n, P_x). *[3]*

b) Let

$$u(x) \;=\; E_x\left[\sum_{n=0}^{\infty} 1_{\{X_n=0\}}\right], \qquad x \in S.$$

Give an intuitive interpretation for $u(x)$, and prove that u satisfies

$$u(x) \;=\; (pu)(x) \;+\; 1_{\{0\}}(x) \qquad \text{for all } x \in S.$$

[7]

Solution. a) For all $n \geq 0$ and all $\mathcal{F}_\infty$-measurable functions $F : S^\infty \to [0, \infty)$, it holds

$$E[F(X_n, X_{n+1}, \ldots) \mid X_{0:n}] \quad = \quad E_{X_n}[F] \qquad P\text{-almost surely.}$$

b) $u(x)$ is the average number of visits of 0 by the process (X_n).
By conditioning on X_1 and using the Markov property, we obtain that

$$\begin{aligned} u(x) \;&=\; E_x\left[E_x\left[\sum_{n=0}^{\infty} 1_{\{X_n=0\}}\middle| X_1\right]\right] \;=\; E_x\left[E_x\left[\sum_{n=1}^{\infty} 1_{\{X_n=0\}}\middle| X_1\right]\right] + 1_0(x) \\ &=\; E_x\left[E_{X_1}\left[\sum_{n=0}^{\infty} 1_{\{X_n=0\}}\right]\right] + 1_0(x) \;=\; E_x\left[u(X_1)\right] + 1_0(x) \;=\; (pu)(x) \;+\; 1_0(x). \end{aligned}$$

3. (Brownian motion [30 points])

a) State the definition of Brownian motion. Show that a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ starting at 0 is a Gaussian process with

$$E[B_t] \;=\; 0 \qquad \text{and} \qquad \mathrm{Cov}[B_s,B_t] \;=\; \min(s,t) \qquad \text{for all } s,t\geq 0.$$

[10]

b) Compute the expectation and the variance of the random variable

$$Z \;:=\; \int_0^1 B_t\,dt.$$

Can you determine the law of Z? *[10]*

c) Sketch the Wiener-Lévy construction of Brownian motion (one page maximum). Mention **in a few keywords** why and in which sense the series expansion converges, and how one verifies that the limit is a Brownian motion. *[10]*

Solution. a) Let $a\in\mathbb{R}^d$. A continuous-time stochastic process $B_t:\Omega\to\mathbb{R}^d$, $t\geq 0$, defined on a probability space $(\Omega,\mathcal{A},P)$, is called a *Brownian motion (starting in a)* iff

- $B_0(\omega) \;=\; a \quad$ for each $\omega\in\Omega$,
- For any partition $0\leq t_0<t_1<\ldots<t_n$, the increments $B_{t_{i+1}}-B_{t_i}$ are independent random variables with distribution

$$B_{t_{i+1}}-B_{t_i} \quad\sim\quad N(0,(t_{i+1}-t_i)I_d),$$

- P-almost every sample path $t\mapsto B_t(\omega)$ is continuous.

For a Brownian motion (B_t) and $0=t_0<t_1<\ldots<t_n$,

$$(B_{t_1}-B_{t_0},\ldots,B_{t_n}-B_{t_{n-1}}) \quad\sim\quad \bigotimes_{i=1}^n N(0,t_i-t_{i-1}),$$

which is a multinormal distribution. Since $B_{t_0}=B_0=0$, we see that

$$\begin{pmatrix} B_{t_1}\\ \vdots \\ B_{t_n}\end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0\\ 1 & 1 & 0 & \ldots & 0 & 0\\ & & \ddots & & & \\ & & & \ddots & & \\ 1 & 1 & 1 & \ldots & 1 & 0\\ 1 & 1 & 1 & \ldots & 1 & 1\end{pmatrix} \begin{pmatrix} B_{t_1}-B_{t_0}\\ \vdots \\ B_{t_n}-B_{t_{n-1}}\end{pmatrix}$$

also has a multivariate normal distribution, i.e., (B_t) is a Gaussian process. Moreover, since $B_t=B_t-B_0$, we have $E[B_t]=0$ and, by independence of the increments,

$$\mathrm{Cov}[B_s,B_t] \;=\; \mathrm{Cov}[B_s,B_s]+\mathrm{Cov}[B_s,B_t-B_s] \;=\; \mathrm{Var}[B_s] \;=\; s \;=\; \min(s,t) \quad \forall\, 0\leq s\leq t.$$

b) Using Fubini's theorem and part a) we obtain that $E[Z] = \int_0^1 E[B_s]ds = 0$. Fubini's theorem is applicable, since $E[\int_0^1 |B_s|ds] = \int_0^1 E[|B_s|]ds \leq \int_0^1 E[B_s^2]^{1/2}ds \leq 1 < \infty$. Again, by Fubini's theorem, we observe that

$$\mathrm{Var}[Z] = E[Z^2] = E\left[\int_0^1 \int_0^1 B_s B_t \, dsdt\right] = \int_0^1 \int_0^1 E[B_s B_t]dsdt = \int_0^1 \int_0^1 \min(s,t)dsdt = \frac{1}{3}.$$

Finally, using Riemann-sums, we observe that almost surely, by continuity of (B_t),

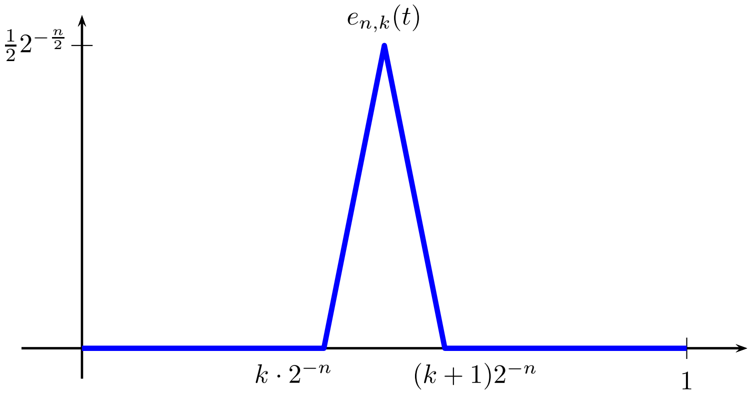
$$Z(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} B_{\frac{k}{n}}(\omega).$$

Since (B_t) is a Gaussian process, $\frac{1}{n}\sum_{k=1}^{n} B_{\frac{k}{n}}$ is normally distributed for all n. Hence, as an a.s. limit of normal random variables, Z is normally distributed, i.e., $Z \sim N(0, 1/3)$.

c) A Brownian motion (B_t) can be obtained as the limit of the series

$$B_t(\omega) \quad = \quad Z(\omega)t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} Z_{n,k}(\omega)e_{n,k}(t), \qquad t \in [0,1], \tag{1}$$

where Z and $Z_{n,k}$, $(n \geq 0, 0 \leq k \leq 2^n-1)$ are independent random variables with standard normal distribution, and $e_{n,k}(t)$ are the Schauder functions.



For P-almost every $\omega \in \Omega$, the series converges uniformly for $t \in [0,1]$. The proof relies on a combination of the Borel-Cantelli Lemma and the Weierstrass criterion for uniform convergence of series of functions. Via the Borel-Cantelli Lemma one first shows that

$$\sup_{t\in[0,1]} \left| \sum_{k=0}^{2^n-1} Z_{n,k}(\omega)e_{n,k}(t) \right| \tag{2}$$

is summable in n. Hence, by the Weierstrass criterion, the partial sums

$$B_t^{(m)}(\omega) \quad = \quad Z(\omega)e(t) + \sum_{n=0}^{m} \sum_{k=0}^{2^n-1} Z_{n,k}(\omega)e_{n,k}(t), \qquad m \in \mathbb{N},$$

converge almost surely uniformly on $[0,1]$. To show that the limit process (B_t) is a Brownian motion, one verifies that (B_t) is a Gaussian process with

$$E[B_t] \;=\; 0 \qquad \text{and} \qquad \mathrm{Cov}[B_s, B_t] \;=\; \min(s,t) \qquad \text{for all } s,t \geq 0.$$

These facts follow from Parseval's relation and since limits of Gaussian random variables are Gaussian.

4. (Martingales and branching processes [25 points])

We consider the branching process defined recursively by

$$Z_0 \;=\; 1, \qquad Z_n \;=\; \sum_{i=1}^{Z_{n-1}} N_{i,n} \quad \text{for } n \geq 1,$$

where $N_{i,n}$ $(i, n \in \mathbb{N})$ are i.i.d. random variables with

$$P[N_{i,n} = 0] = 1/4, \quad P[N_{i,n} = 1] = 1/2, \quad P[N_{i,n} = 2] = 1/4.$$

a) State the definition of a martingale in discrete time, and verify that $(Z_n)_{n\geq 0}$ is a martingale. *[10]*

b) State Doob's upcrossing inequality *(without proof, but including necessary definitions).* *[5]*

c) Prove that almost surely, the sequence (Z_n) converges as $n \to \infty$, and identify the limit Z_∞ *(one page maximum; the upcrossing inequality may be assumed without proof).* *[10]*

Solution. a) A sequence of real-valued random variables $M_n : \Omega \to \mathbb{R}$ $\quad (n = 0, 1, \ldots)$ on the probability space $(\Omega, \mathcal{A}, P)$ is called a *martingale w.r.t. the filtration* $(\mathcal{F}_n)$ iff

- (M_n) is adapted w.r.t. $(\mathcal{F}_n)$,
- M_n is integrable for any $n \geq 0$, and
- $E[M_n \mid \mathcal{F}_{n-1}] \;=\; M_{n-1}$ for any $n \in \mathbb{N}$.

The branching process (Z_n) is adapted w.r.t. the filtration $\mathcal{F}_n = \sigma(N_{i,k} : i \in \mathbb{N}, k \leq n)$. Moreover, Z_n is non-negative for all n, and for P-almost every ω,

$$E[Z_n \mid \mathcal{F}_{n-1}](\omega) \;=\; E\left[\sum_{i=1}^{Z_{n-1}} N_{i,n} \,\middle|\, \mathcal{F}_{n-1}\right](\omega) = \sum_{i=1}^{Z_{n-1}(\omega)} E[N_{i,n}] = Z_{n-1}(\omega).$$

Here we have used that $N_{i,n}$ is independent of $\mathcal{F}_{n-1}$ with $E[N_{i,n}] = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 = 1$. In particular, $E[Z_n] = E[E[Z_n|\mathcal{F}_{n-1}]] = E[Z_{n-1}]$ and thus by induction, $E[Z_n] = 1$ for all n. Since $Z_n \geq 0$ this implies integrability.

b) For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$ we define the number $U_n^{(a,b)}$ of upcrossings over the interval (a,b) before time n by

$$U_n^{(a,b)} \;=\; \max\big\{k \geq 0 : \exists\, 0 \leq s_1 < t_1 < s_2 < t_2 \ldots < s_k < t_k \leq n : Z_{s_i} \leq a, Z_{t_i} \geq b\big\}.$$

Lemma 1 (Doob) *If (Z_n) is a supermartingale then*

$$(b-a)\cdot E[U_n^{(a,b)}] \;\le\; E[(Z_n-a)^-] \qquad \textit{for any } a<b \textit{ and } n\ge 0.$$

c) Let

$$U^{(a,b)} \;=\; \sup_{n\in\mathbb{N}} U_n^{(a,b)}$$

denote the total number of upcrossings of the (super)martingale (Z_n) over an interval (a,b) with $-\infty<a<b<\infty$. By the upcrossing inequality and monotone convergence,

$$E[U^{(a,b)}] \;=\; \lim_{n\to\infty} E[U_n^{(a,b)}] \;\le\; \frac{1}{b-a}\cdot\sup_{n\in\mathbb{N}} E[(Z_n-a)^-]. \tag{3}$$

The right hand side of (3) is finite since $Z_n\ge 0$ implies $a-Z_n\le a$ for all n. Therefore,

$$U^{(a,b)} \;<\; \infty \qquad P\text{-almost surely,}$$

and hence the event

$$\{\liminf Z_n \;\ne\; \limsup Z_n\} \;=\; \bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \{U^{(a,b)}=\infty\}$$

has probability zero. This proves almost sure convergence. Moreover, the limit Z_∞ is non-negative and almost surely finite, since by Fatou's lemma,

$$E[\,Z_\infty\,] \;=\; E[\liminf Z_n] \;\le\; \liminf E[Z_n] \;=1\;<\;\infty.$$

Finally, we note that since $Z_n(\omega)$ is integer-valued, convergence to $Z_\infty(\omega)$ can only occur if there exists $n_0(\omega)$ such that $Z_n(\omega)=Z_\infty(\omega)$ for all $n\ge n_0(\omega)$. Since

$$P[Z_n=k \text{ eventually}]=0 \qquad \text{for any } k\in\mathbb{N},$$

we conclude that $Z_\infty=0$ P-almost surely.