Institut für Angewandte Mathematik
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"Stochastic Processes", Problem Sheet 9.

Please hand in your solutions by Tuesday, 10 December.

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1. (Probabilities of Brownian motion). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ with $B_0 = 0$.

a) Compute the probabilities of the following events:

(i) $B_2 > 2$.

(ii) $B_2 > B_1$.

(iii) $B_2 > B_1 > B_3$.

b) Let $Z := \sup_{t\geq 0} B_t$. Show that λZ has the same distribution as Z for any $\lambda > 0$. Deduce that $Z = +\infty$ $\mathbb{P}$–a.s.

2. (Martingales of Brownian motion). State the definition of a martingale in continuous time. Show that the following processes are martingales:

a) A one-dimensional Brownian motion $(B_t)_{t\geq 0}$ w.r.t. $\mathcal{F}_t = \sigma(B_r : 0 \leq r \leq t)$.

b) $M_t^\lambda = \exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$, $\lambda \in \mathbb{R}$, w.r.t. the same filtration.

c) $h(B_t)$, if $(B_t)_{t\geq 0}$ is a d–dimensional Brownian motion, h is a harmonic function on $\mathbb{R}^d$ such that $h(B_t) \in \mathcal{L}^1(\mathbb{P})$ for all $t \geq 0$, and $\mathcal{F}_t = \sigma(B_r^1, B_r^2, \ldots, B_r^d : 0 \leq r \leq t)$.

3. (Zeros of Brownian paths). Let $(B_t)_{t\in[0,1]}$ be a one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $B_0 = 0$ and all the sample paths $t \mapsto B_t(\omega)$ are continuous.

a) Show that $(t, \omega) \mapsto B_t(\omega)$ is measurable as a map from $\Omega \times [0, 1]$ to $\mathbb{R}$.

b) Compute the expectation and the variance of $\int_0^1 B_s(\omega)\, ds$.

c) Show that $\mathbb{P}$–a.s.: $\lambda\left[\{t \in [0, 1] : B_t = 0\}\right] = 0$.

4. (Star Trek 2: Captain's Log). Mr Spock and Chief Engineer Scott have modified the control system so that the Enterprise is confined to move for ever in a fixed plane passing through the Sun. However, the next 'hop-length' is now automatically set to be the current distance to the Sun ('next' and 'current' being updated in the obvious way). Spock is muttering something about logarithms and random walks, but I wonder whether it is (almost) certain that we will get into the Solar System sometime ...

Hint for Exercise 4: Let $X_n := \log R_n - \log R_{n-1}$*. Prove that* $(X_n)_{n\in\mathbb{N}}$ *is an i.i.d. sequence of random variables each of mean* 0 *and variance* σ^2 *(say), where* $\sigma > 0$*. Let*

$$S_n \;:=\; X_1 + X_2 + \cdots + X_n \,.$$

Prove that if α *is a fixed positive number, then*

$$\mathbb{P}[\inf S_n = -\infty] \;\geq\; \limsup \mathbb{P}[S_n \leq -\alpha\sigma\sqrt{n}] \;>\; 0\,.$$

Why is this sufficient to conclude?