Institut für Angewandte Mathematik
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“Stochastic Processes”, Problem Sheet 8.

Please hand in your solutions by Tuesday, 3 December.

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1. (Hitting times for the 2-dimensional random walk).
Let Z_n be the random walk on $\mathbb{Z}^2$ starting in z_0 and making a step in one of the four directions with equal probability.

a) Show that $|Z_n|^2 - n$ is a martingale.

b) For $r > |z_0|$ let
$$T \;=\; \inf\{n \geq 0 \colon |Z_n| \geq r\}$$
be the exit time from the circle around 0 with radius r. Prove that
$$r^2 - |z_0|^2 \;\leq\; E[T] \;\leq\; (r+1)^2 - |z_0|^2 \,.$$

2. (Random signs).
Let (a_n) be a sequence of real numbers with $\sum a_n^2 = \infty$, and let
$$M_n \;=\; \sum_{k=1}^{n} \varepsilon_k a_k \,, \qquad \varepsilon_k \text{ i.i.d. with } \; P[\varepsilon_k = \pm 1] = 1/2.$$

a) Determine the conditional variance process $\langle M \rangle_n$.

b) For $c > 0$ let $T_c = \inf\{n \geq 0 \colon |M_n| \geq c\}$. Show that $\mathbb{P}[T_c < \infty] = 1$.

c) Conclude that almost surely, the process (M_n) has unbounded oscillations.

3. (Bounds for random walks and bin packing).

a) Let $(S_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}$, i.e. $S_n = U_1 + \cdots + U_n$, where the r.v.'s U_i are i.i.d. with $\mathbb{P}[U_i = 1] = p$ and $\mathbb{P}[U_i = -1] = 1 - p = q$, $p \in (0, 1/2)$. Show that
$$\mathbb{P}\left[\sup_{n\geq 0} S_n \geq k\right] \leq \left(\frac{p}{q}\right)^k \qquad \text{and} \qquad \mathbb{E}\left[\sup_{n\geq 0} S_n\right] \leq \frac{p}{q-p}.$$

b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables taking values in $[0,1]$. How many bins of size 1 are needed to pack n objects of sizes $X_1, X_2, \ldots, X_n$? Let B_n be the minimal number of bins and set
$$M_k \;=\; \mathbb{E}[B_n \mid \sigma(X_1, \ldots, X_k)] \,, \qquad 0 \leq k \leq n \,.$$
Show that $|M_k - M_{k-1}| \leq 1$ and conclude that
$$\mathbb{P}[|B_n - \mathbb{E}[B_n]| \geq \varepsilon] \;\leq\; 2 \cdot e^{-\frac{\varepsilon^2}{2n}} \,.$$
Remark: One can show that asymptotically, $\mathbb{E}[B_n]$ *grows linearly in* n.

4. (CRR model of stock market).

Suppose that in the time interval $(n-1,n)$, an investor holds Φ_n units of an asset with price S_n per unit at time n. We assume that (S_n) is an adapted and (Φ_n) is a predictable stochastic process w.r.t. a filtration $(\mathcal{F}_n)$. If the investor always puts his remaining capital onto a bank account with guaranteed interest rate r ("riskless asset") then the change of his capital V_n during the time interval $(n-1,n)$ is given by

$$V_n \;=\; V_{n-1} + \Phi_n \cdot (S_n - S_{n-1}) + (V_{n-1} - \Phi_n \cdot S_{n-1}) \cdot r. \tag{1}$$

Considering the discounted quantity $\widetilde{V}_n \;=\; V_n/(1+r)^n$, we obtain the equivalent recursion

$$\widetilde{V}_n \;=\; \widetilde{V}_{n-1} + \Phi_n \cdot (\widetilde{S}_n - \widetilde{S}_{n-1}) \qquad \text{for any } n \geq 1. \tag{2}$$

In fact, (1) holds if and only if

$$V_n - (1+r)V_{n-1} \;=\; \Phi_n \cdot (S_n - (1+r)S_{n-1}),$$

which is equivalent to (2). Therefore, the discounted capital at time n is given by

$$\widetilde{V}_n \;=\; V_0 + (\Phi_\bullet \widetilde{S})_n.$$

Thus if the discounted price process $(\widetilde{S}_n)$ is an $(\mathcal{F}_n)$ martingale w.r.t. a given probability measure, then $(\widetilde{V}_n)$ is a martingale as well. In this case, assuming that V_0 is constant, we obtain in particular

$$\mathbb{E}[\widetilde{V}_n] \;=\; V_0,$$

or, equivalently,

$$\mathbb{E}[V_n] \;=\; (1+r)^n V_0 \qquad \text{for any } n \geq 0. \tag{3}$$

This fact, together with the existence of a martingale measure, can now be used for option pricing under a *no-arbitrage assumption*. To this end we assume that the payoff of an option at time N is given by an $(\mathcal{F}_N)$-measurable random variable F. For example, the payoff of a European call option with strike price K based on the asset with price process (S_n) is $S_N - K$ if the price S_n at maturity exceeds K, and 0 otherwise, i.e.,

$$F \;=\; (S_N - K)^+.$$

Suppose further that the option can be *replicated by a hedging strategy* (Φ_n), i.e. there exists an $\mathcal{F}_0$-measurable random variable V_0 and a predictable sequence of random variables $(\Phi_n)_{1\leq n\leq N}$ such that

$$F \;=\; V_N$$

is the value at time N of a portfolio with initial value V_0 w.r.t. the trading strategy (Φ_n). Then, assuming the non-existence of arbitrage possibilities, the option price at time 0 has to be V_0, since otherwise one could construct an arbitrage strategy by selling the option and investing money in the stock market with strategy (Φ_n), or conversely. Therefore, if a martingale measure exists (i.e., an underlying probability measure such that the discounted stock price $(\widetilde{S}_n)$ is a martingale), then the no-arbitrage price of the option at time 0 can be computed by (3) where the expectation is taken w.r.t. the martingale measure.

Consider the CRR binomial model, i.e. $\Omega = \{1+a, 1+b\}^N$ with $-1 < a < r < b < \infty$, $X_i(\omega_1, \ldots, \omega_N) = \omega_i$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, and

$$S_n \;=\; S_0 \cdot \prod_{i=1}^{n} X_i, \quad n = 0, 1, \ldots, N,$$

where S_0 is a constant.

a) *Completeness of the CRR model:* Prove that for any function $F\colon \Omega \to \mathbb{R}$ there exists a constant V_0 and a predictable sequence $(\Phi_n)_{1 \le n \le N}$ such that $F = V_N$ where $(V_n)_{1 \le n \le N}$ is defined by (1), or, equivalently,

$$\frac{F}{(1+r)^N} \;=\; \widetilde{V}_N \;=\; V_0 + (\Phi_\bullet \widetilde{S})_N.$$

Hence in the CRR model, any $\mathcal{F}_N$-measurable function F can be replicated by a predictable trading strategy. Market models with this property are called *complete.*

Hint: Prove inductively that for $n = N, N-1, \ldots, 0$, $\tilde{F} = F/(1+r)^N$ can be represented as

$$\tilde{F} = \widetilde{V}_n + \sum_{i=n+1}^{N} \Phi_i \cdot (\widetilde{S}_i - \widetilde{S}_{i-1})$$

with an $\mathcal{F}_n$-measurable function $\widetilde{V}_n$ and a predictable sequence $(\Phi_i)_{n+1 \le i \le N}$.

b) *Option pricing:* Derive a general formula for the no-arbitrage price of an option with payoff function $F\colon \Omega \to \mathbb{R}$ in the CRR model. Compute the no-arbitrage price for a European call option with maturity N and strike K explicitly.