Institut für Angewandte Mathematik
Winter semester 2024/25
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"Stochastic Processes", Problem Sheet 7.

Please hand in your solutions by Tuesday, 26 November.

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1. (Ruin probabilities). A player has 2 Euros, and he wants to make 10 Euros out of it as fast as possible. He plays a game with the following rules: In each round a fair coin is tossed. If the coin comes up heads, the player will win a sum as high as his stake, and he will get his stake back. Otherwise he will lose his stake. The player decides to choose a fixed strategy: If he owns less than 5 Euros, he bets his whole capital; otherwise he bets exactly the amount needed to reach 10 Euros in case he wins. Show that the player achieves his goal with probability 1/5.

2. (Ruin problem for the asymmetric random walk). Let $p \in (0,1)$ with $p \neq 1/2$. We consider the random walk $S_n = Y_1 + \cdots + Y_n$, Y_i $(i \geq 1)$ i.i.d. with $\mathbb{P}[Y_i = +1] = p$ and $\mathbb{P}[Y_i = -1] = q := 1 - p$.

a) Show that the following processes are martingales:

$$M_n := (q/p)^{S_n}, \qquad N_n := S_n - n(p-q).$$

b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ let $T := \min\{n \geq 0 : S_n \notin (a,b)\}$. Deduce from a) that

$$\mathbb{P}[S_T = a] = \frac{1-(p/q)^b}{1-(p/q)^{b-a}}, \quad \text{and} \quad \mathbb{E}[T] = \frac{b}{p-q} - \frac{b-a}{p-q} \cdot \frac{1-(p/q)^b}{1-(p/q)^{b-a}}.$$

3. (Star Trek I). The control system on the starship *Enterprise* has gone wonky. All that one can do is to set a distance to be travelled. The spaceship will then move that distance in a randomly chosen direction, then stop. The objective is to get into the solar system, a ball of radius r. Initially, the *Enterprise* is at a distance R_0 $(> r)$ from the sun. Let R_n be the distance from sun to *Enterprise* after n 'space-hops'.

a) Show that, for any strategy which always sets a distance not greater than that from the sun to the *Enterprise*, $1/R_n$ is a martingale.
Hint: Use the mean-value property of functions f with $\Delta f = 0$.

b) Deduce that $\mathbb{P}[\textit{Enterprise} \text{ gets into the solar system}] \leq r/R_0$.

c) For each $\varepsilon > 0$, you can choose a strategy which makes this probability greater than $r/R_0 - \varepsilon$. What kind of strategy will that be?

4. (Martingales). Show that:

a) A predictable martingale is almost surely constant.

b) For a nonnegative martingale $(X_n)_{n\geq 0}$ we have almost surely:

$$X_n(\omega) = 0 \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \text{ for all } k \geq 0 \; .$$