Institut für Angewandte Mathematik
Winter semester 2024/25
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"Stochastic Processes", Problem Sheet 6.

Please hand in your solutions by Tuesday, 19 November.

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1. (Stopping times for random walks). Let $p \in (0,1)$, and suppose that $(X_n, \mathbb{P}_x)$ is the canonical time-homogeneous Markov chain with state space $S = \mathbb{Z}$, transition probabilities $p(x, x+1) = p$ and $p(x, x-1) = 1-p$, and $\mathbb{P}_x[X_0 = x] = 1$.

a) Prove or disprove that the following random times are stopping times with respect to the filtration $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

$$\begin{aligned} T_1 &= \min\{n \geq 1 : X_n/n < 2p - 1 + \varepsilon\}, \qquad \text{where } \varepsilon > 0, \\ T_2 &= \sup\{n \geq 0 : X_n = 0\}. \end{aligned}$$

b) Are T_1 and T_2 almost surely finite?

c) State and prove a version of the strong Markov property for $(X_n, \mathbb{P}_x)$.

d) Show that the strong Markov property does not necessarily hold if the random time is not a stopping time.

2. (First step analysis). We consider the random walk on $\mathbb{Z}$ with transition probabilities $p(x, x+1) = p$ and $p(x, x-1) = q := 1-p$ where $p \in (\frac{1}{2}, 1)$. Let

$$u(x) := \mathbb{E}_x\left[\sum_{n=0}^{\infty} a^{X_n}\right], \qquad a > 0.$$

a) Show that $u(x+1) = a \cdot u(x)$.

b) Compute $u(0)$ by conditioning on the first step, and interpret the result.

3. (Distribution of the first return time). Let $(X_n, \mathbb{P}_x)$ be a Markov chain on a countable state space S, and let $T_x := \min\{n \geq 1 : X_n = x\}$. The generating function of the distribution of T_x when starting in x is

$$G(z) = \mathbb{E}_x\left[z^{T_x}\right] \qquad (\,|z| < 1\,).$$

a) Show that

$$\sum_{n=0}^{\infty} \mathbb{P}_x[X_n = x]\, z^n \;=\; \sum_{k=0}^{\infty} \mathbb{E}_x\left[z^{T^{(k)}}\right] \;=\; \frac{1}{1-G(z)}$$

where $T^{(k)}$ is the k-th return time to x.

b) Deduce that for the simple symmetric random walk on $\mathbb{Z}$ we have

$$\frac{1}{1-G(z)} \;=\; \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} z^{2n} \;=\; \frac{1}{\sqrt{1-z^2}}\,,$$

hence $G(z) = 1 - \sqrt{1-z^2}$. In particular $\mathbb{E}_x[T_x] = \infty$.

4. (Extinction probabilities for Birth-and-death chains). Let $(X_n, \mathbb{P}_x)$ be the canonical Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities

$$p(x, x+1) = p_x,\;\; p(x,x) = r_x,\;\; p(x, x-1) = q_x,\;\; \text{and } p(x,y) = 0 \text{ otherwise,}$$

where $p_x + q_x + r_x = 1,\;\; q_0 = 0,\;$ and $p_x, q_x \neq 0\; \forall\, x \neq 0$.

a) Determine all harmonic functions for the Markov chain.
Hint: Deduce from the mean-value property

$$p_x u(x+1) + r_x u(x) + q_x u(x-1) \;=\; u(x) \qquad \forall\, x \geq 1$$

an equivalent equation for the differences $v(x) := u(x+1) - u(x)$.

b) Show that for $0 \leq a < b$,

$$\mathbb{P}_x\left[X_{T_{a,b}} = a\right] \;=\; \frac{h(b) - h(x)}{h(b) - h(a)} \qquad \forall\, a \leq x \leq b\,,$$

where $T_{a,b} = \inf\{n \geq 0 \,:\, X_n \notin (a,b)\,\}$, and

$$h(x) \;=\; \sum_{y=0}^{x-1} \prod_{z=1}^{y} \frac{q_z}{p_z}.$$

c) Compute the extinction probability $\mathbb{P}_x[\exists\, n \geq 0 : X_n = 0]$ when starting in x. Under which condition does the process become extinct almost surely? What happens asymptotically in the other cases?