Institut für Angewandte Mathematik
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"Stochastic Processes", Problem Sheet 5.

Please hand in your solutions by Tuesday, 12 November.

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1. (Thinning of Poisson point processes). Suppose that $(S, \mathcal{B}, \nu)$ is a finite measure space, and $\alpha : S \to [0,1]$ is a measurable function. Let $Z, X_1, X_2, \ldots$, and $U_1, U_2, \ldots$ be independent random variables with distributions $Z \sim \text{Poisson}(\nu(S))$, $X_i \sim \nu/\nu(S)$, and $U_i \sim \text{Unif}(0,1)$. Show that

$$N_\alpha = \sum_{i=1}^{Z} \mathbf{1}_{\{U_i \le \alpha(X_i)\}} \, \delta_{X_i}$$

is a Poisson point process with intensity measure $\alpha(x)\,\nu(dx)$.

2. (Examples of Markov chains). A dice is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

a) The largest number M_n shown up until the nth roll.

b) The number N_n of sixes in n rolls.

c) At time r, the time C_r since the most recent six.

d) At time r, the time B_r until the next six.

3. (Stochastic processes constructed from Bernoulli random variables). Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with

$$\mathbb{P}[\beta_n = 1] = p, \quad \mathbb{P}[\beta_n = 0] = q, \text{ where } p + q = 1.$$

a) We define the stochastic process (X_n) for $n = 2, 3, \ldots$ by

$$X_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{if } \beta_{n-1} = 1, \beta_n = 0, \\ 2 & \text{if } \beta_{n-1} = 0, \beta_n = 1, \\ 3 & \text{if } \beta_{n-1} = \beta_n = 0. \end{cases}$$

(i) Prove that the process $(X_n)_{n\ge2}$ is a Markov chain.

(ii) Compute the transition matrix P.

(iii) Compute the probability $\mathbb{P}[X_{n+3} = 0 \,|\, X_n = 0]$.

b) We define the stochastic process (Y_n) for $n = 2, 3, \ldots$ by

$$Y_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Show that the process $(Y_n)_{n \geq 2}$ is not a Markov chain.

4. (A Markov chain on $\{1, 2, 3\}$). We consider the Markov chain $(X_n)_{n=0,1,2,\cdots}$ with state space $S = \{1, 2, 3\}$, initial state $X_0 = 2$, and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & 1-p-q & q \\ 0 & 0 & 1 \end{pmatrix}, \qquad p, q > 0, \; p + q < 1.$$

a) Show that (X_n) first changes its value at a random time $T \geq 1$ whose law is geometric.

b) Show also that X_T is independent of T, and give the law of X_T.

c) Finally show that $X_t = X_T$ almost surely for $t \geq T$.

5. (Time-space process and process with memory). Suppose that $(X_n)_{n \geq 0}$ is a Markov chain with state space $(S, \mathcal{B})$ and transition kernels $p_n(x, dy)$.

a) Show that the following processes are again Markov chains and identify the state spaces and the transition kernels.

   (i) The time-space process $\hat{X}_n = (n, X_n)$.

   (ii) The process $Z_n = (X_{n-1}, X_n)$, $n \geq 1$.

b) Now suppose that $(X_n)_{n \geq 0}$ is a symmetric random walk on the discrete circle $S = \mathbb{Z}/(k\mathbb{Z})$ where k is odd. Show that as $n \to \infty$, the law of Z_n converges to a unique stationary distribution μ, and identify μ explicitly.