Institut für Angewandte Mathematik
Winter semester 2024/25
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"Stochastic Processes", Problem Sheet 4.

Please hand in your solutions by Tuesday, 5 November.

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1. (Poissonian bears).

At time $t = 0$ there aren't any bears in a village. Brown bears und grizzly bears arrive as independent Poisson processes B and G with intensities β and γ.

a) Show that with probability $\beta/(\beta + \gamma)$ the bear arriving first is a brown bear.

b) Determine the probability that between two succeeding brown bears exactly r grizzly bears arrive in the village.

c) Compute the conditional expectation of the arrival time of the first bear given that $B_1 = 1$.

2. (Poissonian forest).

a) Show that if N and $\tilde{N}$ are independent Poisson point processes on a measurable space $(S, \mathcal{S})$ with intensity measures ν, $\tilde{\nu}$ respectively, then $N + \tilde{N}$ is a Poisson point process with intensity $\nu + \tilde{\nu}$.

b) Let N be a Poisson point process on $\mathbb{R}^2$ with homogeneous intensity measure $\lambda\,\mathrm{d}x$, $\lambda > 0$, and let $R_{(1)} < R_{(2)} < \dots$ be the increasingly ordered distances from the points of the Poisson process to the origin. Show that $R_{(1)}^2, R_{(2)}^2, \dots$ are the points of a Poisson process on $[0, \infty)$ with intensity $\pi\lambda$.

c) Prove directly or with the help of a), that the density of $R_{(k)}$ has the following form:

$$f(r) = \frac{2\pi\lambda r(\lambda\pi r^2)^{k-1} e^{-\lambda\pi r^2}}{(k-1)!}, \quad r > 0.$$

3. (Limit theorems for conditional expectations).

State and prove:

a) A monotone convergence theorem for conditional expectations.

b) Fatou's lemma for conditional expectations.

c) A dominated convergence theorem (Lebesgue's theorem) for conditional expectations.

4. (Simulation of Poisson processes).

Generate simulations (using a language/software of your choice) of

a) Poisson point processes on $[0,1]^2$ with intensity measures $\lambda \, \mathrm{d}x$, $\lambda \in \mathbb{R}_+$,

b) Poisson processes on $[0,t]$ with intensities $\lambda > 0$.

Visualise your results, including variations of the parameters.