Institut für Angewandte Mathematik
Winter semester 2024/25
Andreas Eberle, Francis Lörler

"Stochastic Processes", Problem Sheet 3.

Please hand in your solutions by Tuesday, 29 October.

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1. (Conditional distributions).

a) The joint density of X and Y is given by $f(x, y) := 1/x,\ 0 \leq y \leq x \leq 1$.

(i) Find regular versions of the conditional distributions of X given Y, and of Y given X.

(ii) Compute $\mathbb{E}[X \mid Y]$ and $\mathbb{E}[Y \mid X]$.

b) Let S, T and U be independent exponentially distributed random variables with parameters λ, μ, ν. Show that $\min(T, U)$ is exponentially distributed with parameter $\mu + \nu$, and compute the probabilities $\mathbb{P}[T < U]$ and $\mathbb{P}[S < T < U]$.

2. (Independence and conditional expectations).
Let X, Y be random variables on a joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that X is integrable, and U is independent from the pair (X, Y).

a) Prove that

$$\mathbb{E}[X \mid Y, U] \;=\; \mathbb{E}[X \mid Y] \qquad \mathbb{P}\text{-almost surely.} \tag{1}$$

b) Give an example to show that (1) does not necessarily hold if one only assumes independence of X and U. Explain this fact intuitively.

3. (Martingales of a simple random walk).
Let $(Y_i)_{i \in \mathbb{N}}$ be a sequence of independent random variables with $\mathbb{P}[Y_i = \pm 1] = \frac{1}{2}$, and let

$$X_n = x + S_n \quad \text{where } S_n = Y_1 + \cdots + Y_n.$$

Show that the following processes are martingales w.r.t. the filtration given by $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ (see Problem Sheet 2, Exercise 3 for the definition):

a) X_n,

b) $M_n = X_n^2 - n$,

c) $M_n^\lambda = e^{\lambda X_n - a(\lambda) n} \qquad$ for any $\lambda \in \mathbb{R}$, where $a(\lambda) = \log \cosh \lambda$.

4. (Inequalities for conditional expectations).
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra.

a) Prove the following generalization of Markov's Inequality:

$$\mathbb{P}\big[|X| \geq \alpha \mid \mathcal{G}\big] \;\leq\; \frac{1}{\alpha^k}\mathbb{E}\big[|X|^k \mid \mathcal{G}\big] \quad \mathbb{P}\text{-a.s.} \;\; \text{for any } \alpha > 0.$$

b) State and prove a Cauchy-Schwarz inequality for conditional expectations.