**Institut für Angewandte Mathematik**
**Winter semester 2024/25**
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# "Stochastic Processes", Problem Sheet 13.

**Please hand in your solutions by Wednesday, 22 January.**

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**1. (Martingale convergence I).** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$ on which we consider an $L^2$ bounded martingale $(M_n)_{n\geq 0}$. We define

$$X_n = \sum_{k=1}^{n} \frac{1}{k}(M_k - M_{k-1}).$$

Show that $(X_n)_{n\geq 0}$ is an $(\mathcal{F}_n)$ martingale that converges almost surely and in $L^2$.

**2. (Martingale convergence II).** Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with law $\mathcal{N}(0, \sigma^2)$, $\sigma > 0$. We set $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ and

$$X_n = Y_1 + \ldots + Y_n.$$

a) Show that for every $u \in \mathbb{R}$,

$$Z_n^u = \exp(uX_n - nu^2\sigma^2/2)$$

is an $(\mathcal{F}_n)$ martingale.

b) Conclude that as $n \to \infty$, $Z_n^u$ converges almost surely, and determine the limit $Z_\infty^u$.

c) For which values $u \in \mathbb{R}$ does $Z_n^u$ converge to $Z_\infty^u$ in $L^1$?

**3. (Angle bracket process and martingale convergence).** Suppose that $(M_n)_{n\in\mathbb{Z}_+}$ is a square integrable $(\mathcal{F}_n)$ martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with conditional variance process $\langle M\rangle_n$. We set $\langle M\rangle_\infty := \lim_{n\to\infty}\langle M\rangle_n$.

a) Let $T$ be an $(\mathcal{F}_n)$ stopping time, and let $M_n^T := M_{n\wedge T}$ denote the stopped martingale. Show that almost surely,

$$\langle M^T\rangle_n = \langle M\rangle_{n\wedge T} \qquad \text{for all } n \geq 0.$$

b) Let $a > 0$. Show that $T_a := \inf\{n \geq 0 : \langle M\rangle_{n+1} > a\}$ is an $(\mathcal{F}_n)$ stopping time.

c) Prove that the stopped martingale $(M_n^{T_a})$ converges almost surely and in $L^2$.

d) Hence conclude that $(M_n)$ converges almost surely on the set $\{\langle M\rangle_\infty < \infty\}$.

**4. (Stopped Brownian motion).** Let $(X_t)_{t\geq 0}$ be a one-dimensional Brownian motion started at 0 and let $T = \min\{t : |X_t| = 1\}$ and $T^* = \min\{t : X_t = 1 \text{ or } X_t = -3\}$.

a) Explain why $X_T$ and $T$ are independent random variables.

b) Show that $T^*$ and $X_{T^*}$ are not independent.