Institut für Angewandte Mathematik
Winter semester 2024/25
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"Stochastic Processes", Problem Sheet 12.

Please hand in your solutions by Tuesday, 14 January.

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1. (Ruin probabilities and passage times for Brownian motion).
Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion starting at 0. For $a, b > 0$ let

$$T = \inf\{t \geq 0 \,:\, B_t \notin (-b, a)\} \qquad \text{and} \qquad T_a = \inf\{t \geq 0 \,:\, B_t = a\}$$

denote the first exit time from the interval $(-b, a)$, and the first hitting time of a, respectively. You may assume without proof that both stopping times are almost surely finite. Show that:

a) *Ruin probabilities:* $\mathbb{P}[B_T = a] = b/(a+b), \quad \mathbb{P}[B_T = -b] = a/(a+b)$;

b) *Mean exit time:* $\mathbb{E}[T] = a \cdot b$, and $\mathbb{E}[T_a] = \infty$;

c) *Laplace transform of passage times:* $\mathbb{E}[\exp(-sT_a)] = \exp(-a\sqrt{2s})$ for any $s > 0$;

d) The distribution of T_a on $(0, \infty)$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; a \cdot (2\pi t^3)^{-1/2} \cdot \exp(-a^2/2t).$$

2. (Transition probabilities for continuous-time Markov chains I).

a) Compute $p_t(1, 1)$ for the Markov process with state space $S = \{1, 2, 3\}$ and generator

$$\mathcal{L} \;=\; \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

b) Which of the following matrices are exponentials of Q-matrices ?

$$\text{(i)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. (Transition probabilities for continuous-time Markov chains II). Two fleas are bound together to take part in a nine-legged race on the vertices A, B, C of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with exponential distribution, mean $1/\lambda$. Flea 2 behaves similarly, but hops in the anticlockwise direction, the times between his hops having mean $1/\mu$. Show that the probability that they are at A at a given time $t > 0$ (starting from A at time $t = 0$) is

$$\frac{1}{3} + \frac{2}{3} \exp\left(-\frac{3(\lambda+\mu)t}{2}\right) \cos\left(\frac{\sqrt{3}(\lambda-\mu)t}{2}\right). \tag{1}$$

4. (Law of the iterated logarithm). Let $(B_t)_{t\geq 0}$ be a one dimensional Brownian motion with $B_0 = 0$. Recall from the lectures that almost surely,

$$\limsup_{t\downarrow 0} \frac{B_t}{h(t)} \leq +1, \qquad \text{where} \qquad h(t) = \sqrt{2t \log\log t^{-1}}.$$

Complete the proof of the Law of Iterated Logarithm, i.e., show that almost surely,

$$\limsup_{t\downarrow 0} \frac{B_t}{h(t)} = +1$$

To this end, you may proceed in the following way:

a) Show that almost surely,

$$\liminf_{t\downarrow 0} \frac{B_t}{h(t)} \geq -1.$$

b) Let $\theta \in (0,1)$ and consider the increments $Z_n = B_{\theta^n} - B_{\theta^{n+1}}, n \in \mathbb{N}$. Show that if $\epsilon > 0$, then

$$\mathbb{P}[Z_n > (1-\epsilon)h(\theta^n) \quad \text{infinitely often}] = 1.$$

Hint: $\int_x^\infty \exp(-z^2/2)dz \geq (x^{-1} - x^{-3})\exp(-x^2/2)$.

c) Using the statements in a) and b), conclude that

$$\limsup_{t\searrow 0} \frac{B_t}{h(t)} \geq 1-\epsilon \qquad \mathbb{P}\text{-almost surely for every } \epsilon > 0.$$

Hence complete the proof of the LIL.