Institut für Angewandte Mathematik
Winter semester 2024/25
Andreas Eberle, Francis Lörler

"Stochastic Processes", Problem Sheet 11.

Please hand in your solutions by Tuesday, 7 January.

We wish you a merry Christmas and a happy new year!

1. (Transformations of Brownian motion).

a) Show that the projection of a d-dimensional Brownian motion onto a line through the origin yields a one-dimensional Brownian motion: Suppose that $(W_t^{(1)}, \dots, W_t^{(d)})$ is a d-dimensional Brownian motion starting from 0, and let $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ with $\sum_{i=1}^d \lambda_i^2 = 1$. Show that $X_t = \sum_{i=1}^d \lambda_i W_t^{(i)}$ is a Brownian motion starting from 0.

b) Show that rotating a Brownian motion about the origin yields another Brownian motion: Let W be a d-dimensional Brownian motion starting from 0 and let A be a $d \times d$ orthogonal matrix. Show that $Y_t = AW_t$ is again a d-dimensional Brownian motion.

c) Formulate a statement that includes a) and b) !

2. (Construction of Brownian motion). In the lecture we have shown the existence of a probability measure μ_0 on $C([0,1])$ such that under μ_0, the canonical process (X_t) is a Brownian motion with $X_0 = 0$ almost surely. Apply this result to construct a d-dimensional Brownian motion (B_t) for all times $t \in [0,\infty)$ such that $B_0 = x$ almost surely, where $x \in \mathbb{R}^d$ is a fixed starting point. Write down the probability space and the definition of the random variables explicitly, and verify that (B_t) is indeed a Brownian motion.

3. (Optional stopping for continuous martingales). Suppose that $(\mathcal{F}_t)_{t\in[0,\infty)}$ is a continuous time filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A random variable T with values in $[0,\infty]$ is called an $(\mathcal{F}_t)$ *stopping time* iff $\{T \le t\} \in \mathcal{F}_t$ for any $t \in [0,\infty)$.

a) Let $(M_t)_{t\in[0,\infty)}$ be an $(\mathcal{F}_t)$ martingale with continuous sample paths. Prove that for every $(\mathcal{F}_t)$ stopping time T and for all $t \in [0,\infty)$,

$$\mathbb{E}[M_{T\wedge t}] \;=\; \mathbb{E}[M_0] \,.$$

Hint: Approximate $T \wedge t$ by the discrete stopping times $T_n = \lceil 2^n (T \wedge t)\rceil 2^{-n}$. For simplicity, you may assume that there exists $p > 1$ such that $M_t \in \mathcal{L}^p$ for all $t \ge 0$, and apply Doob's L^p inequality. The proof without this assumption is more involved.

b) Suppose that $u \in C^2(\mathbb{R}^d)$ is a harmonic function and let D be an open bounded subset of $\mathbb{R}^d$. Prove that if $u = f$ on ∂D, then u has the stochastic representation

$$u(x) = \mathbb{E}_x [f(X_T)] , \qquad x \in D$$

where $\mathbb{E}_x$ denotes the expectation w.r.t. Wiener measure with start in x, $(X_t)_{t\in[0,\infty)}$ is the canonical Brownian motion, and

$$T = \inf \{t \geq 0 : X_t \in \partial D\} .$$

4. (Wiener–Lévy representation and quadratic variation).

The quadratic variation $[x]_t$ of a continuous function $x : [0, \infty) \to \mathbb{R}$ along the sequence of dyadic partitions of the intervals $[0, t]$ is defined by

$$[x]_t := \lim_{m\to\infty} \sum_{i=1}^{2^m} \left| x(t_i^{(m)}) - x(t_{i-1}^{(m)}) \right|^2 ; \qquad t_i^{(m)} = i2^{-m}t.$$

a) Show that the quadratic variation of a continuously differentiable function x vanishes, i.e., $[x]_t = 0$ for any $t \geq 0$.

b) Let

$$x(t) = x(1) \cdot t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k} \cdot e_{n,k}(t), \qquad a_{n,k} \in \mathbb{R},$$

be the expansion of a function $x \in C([0, 1])$ with $x(0) = 0$ in the basis of Schauder functions. Show that

$$[x]_1 = \lim_{m\to\infty} \frac{1}{2^m} \sum_{n=0}^{m-1} \sum_{k=0}^{2^n-1} (a_{n,k})^2.$$

c) Deduce that almost every path of Brownian motion has quadratic variation $[B]_t = t$. Why does it suffice to consider $t = 1$?

d) Determine the quadratic variation of the "self-similar" function

$$g(t) := t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} e_{n,k}(t)$$

on the interval $[0, 1]$, and on $[0, t]$ for $t \in [0, 1)$. Compare with the result from c).