Institut für Angewandte Mathematik
Winter semester 2024/25
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"Stochastic Processes", Problem Sheet 10.

Please hand in your solutions by Tuesday, 17 December.

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1. (Symmetries of Brownian motion). Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be independent one-dimensional Brownian motions starting at 0.

a) Show that the following processes are Brownian motions:

$$\text{(i)}\ -X_t \qquad \text{(ii)}\ X_{t+h}-X_h\ \ (h\geq 0 \text{ fixed}) \qquad \text{(iii)}\ \frac{1}{\sqrt{a}}X_{at}\ \ (a>0 \text{ fixed}).$$

b) Show that $B_t := (X_t - Y_t)/\sqrt{2}$ is a Brownian motion.

c) True or false: With probability 1 we have $X_t = Y_t$ for infinitely many $t > 0$.

2. (Wiener-Lévy representation of Brownian motion). For $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, 2^n - 1$, the Schauder functions $e_{n,k} \in C([0,1])$ are defined by

$$\begin{aligned} e_{0,0}(t) &:= \min(t, 1-t), \\ e_{n,k}(t) &:= \begin{cases} 2^{-n/2} e_{0,0}(2^n t - k) & \text{for } t \in [k2^{-n}, (k+1)2^{-n}], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

a) Let $x \in C([0,1])$ with $x(0) = 0$. Show that the sequence

$$x^{(m)}(t) \;=\; x(1)\cdot t + \sum_{n=0}^{m}\sum_{k=0}^{2^n-1} a_{n,k}\cdot e_{n,k}(t), \qquad m \in \mathbb{N},$$

with $a_{n,k} = -2^{n/2}\Delta_{n,k}x$, $\Delta_{n,k}x = x((k+1)2^{-n}) - 2x((k+1/2)2^{-n}) + x(k2^{-n})$, converges to $x(t)$ uniformly for $t \in [0,1]$.

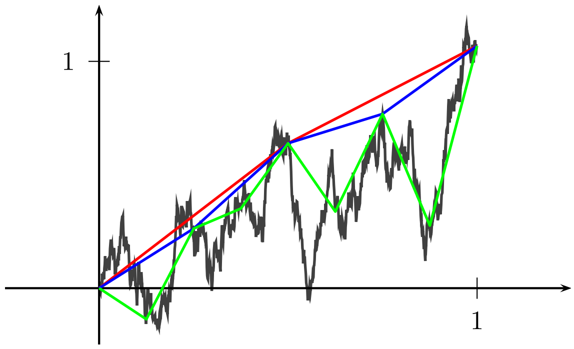
b) Under Wiener measure μ_0 on $\Omega = C([0,1])$, the random variables

$$X_1(\omega) \qquad \text{und} \qquad Z_{n,k}(\omega) \;=\; -2^{n/2}\cdot \Delta_{n,k}X(\omega) \qquad (n \geq 0, 0 \leq k < 2^n),$$

are independent with distribution $\mathcal{N}(0,1)$, and the *Wiener–Lévy representation*

$$X_t(\omega) \;=\; X_1(\omega)\cdot t + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} Z_{n,k}(\omega)\cdot e_{n,k}(t) \qquad \text{holds for all } \omega \in \Omega.$$

c) How can this be used in order to simulate sample paths of Brownian motion ?



3. (Local maxima of Brownian paths). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the following statements hold for almost every ω:

a) The trajectory $t \mapsto B_t(\omega)$ is not monotone in any interval $[a, b]$ with $a < b$.

b) The set of local maxima of $t \mapsto B_t(\omega)$ is dense in $[0, \infty)$.

c) All local maxima of $t \mapsto B_t(\omega)$ are strict (i.e., for any local maximum m there exists an $\varepsilon > 0$ such that $B_t(\omega) < B_m(\omega)$ for all $t \in (m - \varepsilon, m + \varepsilon)$).