Institut für angewandte Mathematik
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„Stochastic Processes", Problem Sheet 2.

Please hand in your solutions until Tuesday, October 22.

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1. (Conditioning and transformations of random variables).

a) Let X be a Poisson distributed random variable with random intensity Λ, where Λ is exponentially distributed with parameter μ. Determine the distribution of X.

b) Let X be a standard normally distributed random variable and let $a > 0$. Show that the random variable Y given by

$$Y := \begin{cases} X & \text{if } |X| < a, \\ -X & \text{if } |X| \geq a, \end{cases}$$

has again the $\mathcal{N}(0,1)$ distribution, and find an expression for $\rho(a) = \mathrm{Cov}[X,Y]$ in terms of the density function φ of X. Does the pair (X,Y) have a bivariate normal distribution?

2. (Properties of conditional expectations).
Let X, Y be non-negative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and let $\mathcal{F} \subseteq \mathcal{A}$ be a σ–algebra. Show that the following assertions are true for any version of the conditional expectations:

a) $\mathbb{E}[\,\mathbb{E}[X|\mathcal{F}]\,] \;=\; \mathbb{E}[X] \qquad \mathbb{P}\text{–a.s.}$

b) If $\sigma(X)$ is independent of $\mathcal{F}$, then

$$\mathbb{E}[X|\mathcal{F}] \;=\; \mathbb{E}[X] \qquad \mathbb{P}\text{–a.s.}$$

c) If Y is $\mathcal{F}$–measurable, then

$$\mathbb{E}[Y \cdot X|\mathcal{F}] \;=\; Y \cdot \mathbb{E}[X|\mathcal{F}] \qquad \mathbb{P}\text{–a.s.}$$

d) $\mathbb{E}[\lambda X + \mu Y|\mathcal{F}] \;=\; \lambda\,\mathbb{E}[X|\mathcal{F}] + \mu\,\mathbb{E}[Y|\mathcal{F}] \qquad \mathbb{P}\text{–a.s. for all } \lambda, \mu \in \mathbb{R}.$

3. (Martingales).
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be an increasing sequence of σ–algebras $\mathcal{F}_n \subseteq \mathcal{A}$. Intuitively, $\mathcal{F}_n$ could describe the "information" on a random experiment that is available at time n. A real-valued stochastic process X_n $(n = 0, 1, 2, \ldots)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is called a *martingale* with respect to $(\mathcal{F}_n)$ if for any $n \geq 0$:

(i) X_n is $\mathcal{F}_n$–measurable and integrable, and

(ii) $\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \;=\; 0 \qquad \mathbb{P}$–a.s. *(„fair game")*

Show that:

a) If Y_i $(i \geq 0)$ are independent and integrable with $\mathbb{E}[Y_i] = 0$, then

$$S_n \;=\; Y_1 + Y_2 + \cdots + Y_n$$

is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

b) For all random variables $X \in \mathcal{L}^1$, the successive prediction values

$$X_n \;=\; \mathbb{E}[X | \mathcal{F}_n]$$

form a martingale.

c) If (X_n) is a martingale with $X_n \in \mathcal{L}^2$, then the increments $\Delta X_n := X_n - X_{n-1}$ are pairwise uncorrelated. State a weak law of large numbers.

4. (Transformations of exponential random variables).
Let T and R be independent exponentially distributed random variables with parameters λ and μ respectively. Determine

a) the conditional distribution of T given $T + R$,

b) the distribution of T/R.

5. (*Additional problem, optional).
Let $X_0, X_1, \ldots : \Omega \to \mathbb{R}$ be i.i.d. random variables with density function f and distribution function F, and let

$$N \;=\; \min\{n \geq 1 : X_n > X_0\}\,.$$

Show that the distribution function of X_N is given by $F + (1 - F)\log(1 - F)$.