Institut für angewandte Mathematik
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„Stochastic Processes", Problem Sheet 1.

Please hand your solutions until Tuesday, October 15.

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1. (Conditional Expectations). Let $X, Y : \Omega \to [0, \infty)$ be independent and identically distributed (iid) discrete random variables with expectation m.

a) Find the mistake in the following reasoning:

$$\mathbb{E}[X \,|\, X + Y = z] \;=\; \mathbb{E}[X \,|\, X = z - Y] \;=\; \mathbb{E}[z - Y] \;=\; z - m.$$

b) Show that

$$\mathbb{E}[X \,|\, X + Y] \;=\; \frac{1}{2}(X + Y).$$

c) Can one proof in a similar way that

$$\mathbb{E}[X \,|\, X \cdot Y] \;=\; (X \cdot Y)^{1/2} \text{ ?}$$

2. (Properties of conditional expectations). Let $Y : \Omega \to S$ be a *discrete* random variable, and let $X : \Omega \to \mathbb{R}$ be an integrable real-valued random variable, both defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Prove in an elementary way (i.e., without referring to the definition of general conditional expectations) that:

a) The map $X \to \mathbb{E}[X|Y]$ is almost surely linear and monotone.

b) If $X = \tilde{X}$ almost surely, then also $\mathbb{E}[X|Y] = \mathbb{E}[\tilde{X}|Y]$ almost surely.

c) For any $f : S \to \mathbb{R}$ such that $f(Y) \cdot X \in \mathcal{L}^1$,

$$\mathbb{E}[f(Y) \cdot X|Y] \;=\; f(Y) \cdot \mathbb{E}[X|Y] \qquad \mathbb{P}\text{-a.s.}$$

Which result do we obtain if X and Y are independent ?

3. (Branching with immigration). Each generation of a branching process (with a single progenitor) is augmented by a random number of immigrants who are indistinguishable from the other members of the population. Suppose that the numbers of immigrants in different generations are independent of each other and of the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function G_n of the size of the nth generation satisfies

$$G_{n+1}(s) \;=\; G_n(G(s))H(s) \;,$$

where G is the probability generating function of a typical family of offspring.

4. (Error Detection). A factory is producing notebooks that are defect with probability p. A test identifies defect notebooks with probability $1-\varepsilon$, whereas non-defect notebooks always pass the test.

a) Show that the probability that a notebook which passed the test is nevertheless defect, is $\varepsilon p/(1-p+\varepsilon p)$.

b) The factory produces n notebooks a day. Let X denote the number of defect notebooks, and let Y be the number of notebooks identified as defect. Under suitable independence assumptions show that

$$\mathbb{E}[X \,|\, Y] \;=\; Y + (n-Y) \cdot \frac{\varepsilon p}{1-p+\varepsilon p} \;=\; \frac{\varepsilon pn + (1-p)Y}{1-p+\varepsilon p}\,.$$

5. (Literature check). Go to the maths library and have a look at textbooks on stochastic processes, for example those recommended below. Borrow a book from the Lehrbuchsammlung (textbook collection).

Recommended textbooks on Stochastic Processes

- Bass: Stochastic processes
- Breiman: Probability
- Durrett: Probability: Theory and Examples
- Grimmett/Stirzaker: Probability and Random Processes
- Kersting/Wakolbinger: Stochastische Prozesse
- Klenke: Probability Theory (also available in German)
- Koralov/Sinai: Theory of Probability and Random Processes
- Le Gall: Brownian motion, martingales and stochaastic calculus
- Norris: Markov Chains
- Rogers/Williams: Diffusions, Markov processes and martingales
- Williams: Probability with Martingales