Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 9.

Please hand in your solutions before 3 pm on Tuesday, June 19.

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1. (Hitting times for the 2-dimensional random walk). Let Z_n be the random walk on $\mathbb{Z}^2$ starting in z_0 and making a step in one of the four directions with equal probability.

a) Show that $|Z_n|^2 - n$ is a martingale.

b) For $r > |z_0|$ let
$$T \ := \ \inf\{n \geq 0 : |Z_n| \geq r\}$$
be the exit time from the circle around 0 with radius r. Prove that
$$r^2 - |z_0|^2 \ \leq \ E[T] \ \leq \ (r+1)^2 - |z_0|^2 \, .$$

2. (Star Trek I). The control system on the star-ship *Enterprise* has gone wonky. All that one can do is to set a distance to be travelled. The spaceship will then move that distance in a randomly chosen direction, then stop. The object is to get into the Solar System, a ball of radius r. Initially, the *Enterprise* is at a distance R_0 ($> r$) from the Sun. Let R_n be the distance from Sun to *Enterprise* after n 'space-hops'.

a) Show that, for any strategy which always sets a distance not greater than that from the Sun to the *Enterprise*, $1/R_n$ is a martingale. *(Hint: Use the mean-value property of harmonic functions (Proof?): If $\Delta f = 0$ on a ball $B \subset \mathbb{R}^3$, then the value of f at the center of the ball is equal to the average of f on the surface ∂B).*

b) Deduce that $P[$*Enterprise* gets into Solar System$] \leq r/R_0$.

c) For each $\varepsilon > 0$, you can choose a strategy which makes this probability greater than $r/R_0 - \varepsilon$. What kind of strategy will that be?

3. (Random signs). Let (a_n) be a sequence of real numbers with $\sum a_n^2 = \infty$, and let
$$M_n \ = \ \sum_{k=1}^{n} \varepsilon_k a_k \, , \qquad \varepsilon_k \text{ i.i.d. with } \ P[\varepsilon_k = \pm 1] = 1/2.$$

a) Determine the conditional variance process $\langle M \rangle_n$.

b) For $c > 0$ let $T_c := \inf\{n \geq 0 : |M_n| \geq c\}$. Show that $P[T_c < \infty] = 1$.

c) Conclude that almost surely, the process (M_n) has unbounded oscillations.

4. (Random walk on a bow tie). A particle performs a random walk on the graph drawn beneath. From any vertex its next step is equally likely to be to any neighbouring vertex. Initially it is at A. Find the expected value of:

a) the time of first return to A, given no prior visit by the particle to E,

b) the number of visits to D before returning to A, given no prior visit to E.