Institut für angewandte Mathematik
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Andreas Eberle, Kaveh Bashiri

UNIVERSITÄT BONN

„Stochastic Processes", Problem Sheet 8.

Please hand in your solutions before 3 pm on Tuesday, June 12.

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1. (Ruin problem for the asymmetric random walk revisited).
Let $p \in (0,1)$ with $p \neq 1/2$. We consider the random walk $S_n = Y_1 + \cdots + Y_n$, Y_i $(i \geq 1)$ i.i.d. with $P[Y_i = +1] = p$ and $P[Y_i = -1] = q := 1-p$.

a) Show that the following processes are martingales:

$$M_n \ := \ (q/p)^{S_n} \ , \qquad N_n \ := \ S_n - n(p-q)\,.$$

b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ let $T := \min\{n \geq 0 : S_n \notin (a,b)\}$. Deduce from a) that

$$P[S_T = a] \ = \ \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,, \quad \text{and} \quad E[T] \ = \ \frac{b}{p-q} - \frac{b-a}{p-q} \cdot \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,.$$

2. (Martingales). Show that:

a) A predictable martingale is almost surely constant.

b) For a nonnegative martingale $(X_n)_{n\geq 0}$ we have almost surely:

$$X_n(\omega) = 0 \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \ \text{ for all } k \geq 0\,.$$

3. (Dichotomy of transience and recurrence). Consider a time homogeneous Markov chain (X_n, P_x) with transition kernel p on a countable set S. For $y \in S$ let

$$B_y(\omega) = \sum_{n=0}^{\infty} 1_{\{y\}}(X_n(\omega))$$

denote the number of visits of the Markov chain to the state y, and let

$$T_y = \min\{n \geq 1 : X_n = y\}$$

denote the first passage/return time to the state y.

a) Prove that $P_x[B_x \geq n] = P_x[T_x < \infty]^n$.

b) Prove the following dichotomy: Either

$$P_x[T_x < \infty] = 1 \quad \text{and} \quad B_x = \infty \ P_x\text{-a.s.} \quad \text{and} \quad G(x,x) = \infty \quad \textbf{(recurrence)},$$

or

$$P_x[T_x < \infty] < 1 \quad \text{and} \quad B_x < \infty \ P_x\text{-a.s.} \quad \text{and} \quad G(x,x) < \infty \quad \textbf{(transience)}.$$

c) Consider a random walk on $\mathbb{Z}^d$ given by

$$X_{n+1} = X_n + U_{n+1},$$

where the increments $U_i,\ i = 1, 2, \dots,$ are independent and uniformly distributed on $\{-1, +1\}^d$. Study recurrence and transience of the random walk depending on the dimension d.

4. (Stationary distributions of autoregressive processes II).
An AR(1) process (X_n) in $\mathbb{R}^d$ is given by a recursion

$$X_n \;=\; \theta X_{n-1} + U_n,$$

where $\theta : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear map, and the random variables U_n are independent and $N(0, I_d)$ distributed.

Prove that: If θ is not symmetric, then under suitable assumptions to be specified, there exists a stationary distribution, but the "Detailed Balance" condition is not satisfied!