Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 7.

Please hand in your solutions before 3 pm on Tuesday, June 5.

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1. (Stationary distributions of autoregressive processes I).
An AR(1) process $(X_n)_{n\in\mathbb{Z}_+}$ is a stochastic process with state space $\mathbb{R}^d$ that is given by a recursion

$$X_n \;=\; \theta X_{n-1} + U_n,$$

where $\theta : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear map, and the random variables U_n are independent and $N(0, I_d)$ distributed.

(i) Show that (X_n) is a Markov chain and determine the transition kernel.

(ii) Now assume $d = 1$. Show that, under an appropriate assumption, there is a normal distribution which is invariant w.r.t. p.

2. (Ruin probabilities). A player has 2 Euros, and he wants to make 10 Euros out of it as fast as possible. He plays a game with the following rules: In each round a fair coin is tossed. If the coin comes up heads, the player will win a sum as high as his stake, and he will get his stake back. Otherwise he will lose his stake. The player decides to choose a fixed strategy: If he owns less than 5 Euros, he bets his whole capital; otherwise he bets exactly the amount needed to reach 10 Euros in case he wins. Show that the player achieves his goal with probability 1/5.

3. (Optional stopping). Let $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration.

a) Prove by induction: If (X_n) is a martingale and T is a stopping time w.r.t. $(\mathcal{F}_n)$ then

$$E[X_{T\wedge n}] \;=\; E[X_0] \qquad \forall\, n \geq 0\,.$$

b) Give sufficient conditions such that

$$E[X_T] \;=\; E[X_0].$$

4. (Extinction probabilities for Birth-and-death chains).
Let (X_n, P_x) be the canonical Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities

$$p(x, x+1) = p_x,\;\; p(x,x) = r_x,\;\; p(x, x-1) = q_x,\;\; \text{and } p(x,y) = 0 \text{ otherwise},$$

where $p_x + q_x + r_x = 1,\;\; q_0 = 0,\;$ and $p_x, q_x \neq 0\;\forall\, x \neq 0.$

a) Deduce from the mean-value property

$$p_x u(x+1) + r_x u(x) + q_x u(x-1) \;=\; u(x) \qquad \forall\, x \geq 1$$

an equivalent equation for the differences $v(x) := u(x+1) - u(x)$. Hence determine all harmonic functions for the Markov chain.

b) Show that for $0 \leq a < b$,

$$P_x\left[X_{T_{a,b}} = a\right] \;=\; \frac{h(b) - h(x)}{h(b) - h(a)} \qquad \forall\, a \leq x \leq b\,,$$

where $T_{a,b} = \inf\{n \geq 0 \,:\, X_n \notin (a,b)\}$, and

$$h(x) \;=\; \sum_{y=0}^{x-1} \prod_{z=1}^{y} \frac{q_z}{p_z}.$$

c) Compute the extinction probability $P_x[\exists\, n \geq 0 \,:\, X_n = 0]$ when starting in x. Under which condition does the process become extinct almost surely? What happens asymptotically in the other cases?