Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 6.

Please hand in your solutions before 3 pm on Tuesday, May 29.

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1. (First step analysis).

We consider the Random Walk on $\mathbb{Z}$ with transition probabilities $p(x, x+1) = p$ and $p(x, x-1) = q := 1 - p$ where $p \in (\frac{1}{2}, 1)$. Let

$$u(x) \;:=\; E_x\left[\sum_{n=0}^{\infty} a^{X_n}\right], \qquad a > 0\,.$$

a) Show that $u(x+1) \;=\; a \cdot u(x)\,.$

b) Compute $u(0)$ by conditioning on the first step, and interpret the result.

2. (Invariant probability measures).

Suppose that $p(x, dy)$ is a stochastic kernel on a measurable state space $(S, \mathcal{B})$, and μ is a positive measure on $(S, \mathcal{B})$ (not necessarily a probability measure). Then μ is called *invariant w.r.t.* p iff $\mu p = \mu$, and μ satisfies the *detailed balance condition w.r.t.* p iff

$$\int\!\!\int \mu(dx)\, p(x, dy)\, f(x, y) \;=\; \int\!\!\int \mu(dy)\, p(y, dx)\, f(x, y) \quad \text{for all measurable } f : S \times S \to \mathbb{R}_+.$$

a) Show that a measure that satisfies the detailed balance condition is invariant.

b) Suppose that μ is an invariant probability measure for p and (X_n, P) is a time-homogeneous Markov chain with initial distribution μ and transition probability p. Show that $X_n \sim \mu$ for all $n \geq 0$.

c) Now let $p \in (0, 1)$, and consider a Markov chain with state space $\mathbb{Z}_+$ and transition probabilities $p(x, x+1) = p$ for $x \geq 0$, $p(x, x-1) = q := 1 - p$ for $x \geq 1$, and $p(0, 0) = q$.

   (i) Find a nontrivial invariant measure.

   (ii) Show that if $p < q$ then there is a unique invariant probability measure.

   (iii) Show that if $p \geq q$ then an invariant probability measure does not exist.

3. (Returns to the starting point).

We consider a time-homogeneous Markov chain on $\{1,2,3\}$ starting in the state x with transition matrix

$$p := \begin{pmatrix} 1-2q & 2q & 0 \\ q & 1-2q & q \\ 0 & 2q & 1-2q \end{pmatrix}.$$

For $x = 1,2,3$ compute

a) the n–step return probabilities $P[X_n = x]$,

b) the average number of returns to the starting point x until time n.

What is the relative frequency of visits to the starting point in the limit as $n \to \infty$?

4. (Distribution of the first return time).

Let (X_n, P_x) be a Markov chain on a countable state space S, and let $T_x := \min\{n \geq 1 : X_n = x\}$. The generating function of the distribution of T_x when starting in x is

$$G(z) \;=\; E_x\left[z^{T_x}\right] \qquad (\,|z| < 1\,).$$

a) Show that:

$$\sum_{n=0}^{\infty} P_x[X_n = x]\, z^n \;=\; \sum_{k=0}^{\infty} E_x\left[z^{T^{(k)}}\right] \;=\; \frac{1}{1-G(z)}$$

where $T^{(k)}$ is the k–th return time to x.

b) Deduce that for the simple random walk on $\mathbb{Z}$ we have

$$\frac{1}{1-G(z)} \;=\; \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} z^{2n} \;=\; \frac{1}{\sqrt{1-z^2}}\,,$$

hence $G(z) = 1 - \sqrt{1-z^2}$. In particular $E_x[T_x] = \infty$.