Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 5.

Please hand in your solutions before 3 pm on Tuesday, May 15.

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1. (Mapping, superposition and thinning of Poisson point processes).
Let ν and $\tilde{\nu}$ be finite measures on a measurable space $(S, \mathcal{B})$, and let $\alpha : S \to [0,1]$ and $\Phi : S \to T$ be measurable functions, where $(T, \mathcal{C})$ is another measurable space.

a) Let $Z, X_1, X_2, \ldots$ be independent random variables with distributions $Z \sim \text{Poisson}(\nu(S))$ and $X_i \sim \nu/\nu(S)$. Show that

$$N = \sum_{i=1}^{Z} \delta_{\Phi(X_i)}$$

is a Poisson point process with intensity measure $\nu \circ \Phi^{-1}$.

b) Show that if N and $\tilde{N}$ are independent Poisson point processes with intensity measures ν, $\tilde{\nu}$ respectively, then $N + \tilde{N}$ is a Poisson point process with intensity $\nu + \tilde{\nu}$.

c) Let $Z, X_1, X_2, \ldots,$ and $U_1, U_2, \ldots$ be independent random variables with distributions $Z \sim \text{Poisson}(\nu(S))$, $X_i \sim \nu/\nu(S)$, and $U_i \sim \text{Unif}(0,1)$. Show that

$$N_\alpha = \sum_{i=1}^{Z} \mathbf{1}_{\{U_i \leq \alpha(X_i)\}} \, \delta_{X_i}$$

is a Poisson point process with intensity measure $\alpha(x)\, \nu(dx)$.

2. (Examples of Markov chains). A dice is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

a) The largest number M_n shown up until the nth roll.

b) The number N_n of sixes in n rolls.

c) At time r, the time C_r since the most recent six.

d) At time r, the time B_r until the next six.

3. (Stochastic processes constructed from Bernoulli random variables). Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables with

$$\mathbb{P}[\beta_n = 1] = p, \quad \mathbb{P}[\beta_n = 0] = q, \text{ where } p + q = 1.$$

a) We define the stochastic process (X_n) for $n = 2, 3, \ldots$ by

$$X_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{if } \beta_{n-1} = 1, \beta_n = 0, \\ 2 & \text{if } \beta_{n-1} = 0, \beta_n = 1, \\ 3 & \text{if } \beta_{n-1} = \beta_n = 0. \end{cases}$$

(i) Prove that the process $(X_n)_{n\geq 2}$ is a Markov chain.

(ii) Compute the transition matrix P.

(iii) Compute the probability $\mathbb{P}[X_{n+3} = 0 \,|\, X_n = 0]$.

b) We define the stochastic process (Y_n) for $n = 2, 3, \ldots$ by

$$Y_n = \begin{cases} 0 & \text{if } \beta_{n-1} = \beta_n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Show that the process $(Y_n)_{n\geq 2}$ is not a Markov chain.

4. (A Markov chain on $\{1, 2, 3\}$). We consider the Markov chain $(X_n)_{n=0,1,2,\cdots}$ with state space $S = \{1, 2, 3\}$, initial state $X_0 = 2$, and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & 1-p-q & q \\ 0 & 0 & 1 \end{pmatrix}, \qquad p, q > 0, \; p + q < 1.$$

a) Show that (X_n) first changes its value at a random time $T \geq 1$ whose law is geometric.

b) Show also that X_T is independent of T, and give the law of X_T.

c) Finally show that $X_t = X_T$ almost surely for $t \geq T$.