Institut für angewandte Mathematik
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„Stochastic Processes", Problem Sheet 2.

Please hand in your solutions before 3 pm on Tuesday, April 24, into the marked post boxes opposite to the maths library.

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1. (Conditioning and transformations of random variables).

a) Let X be a Poisson distributed random variable with random intensity Λ, where Λ is exponentially distributed with parameter μ. Determine the distribution of X.

b) Let X be a standard normally distributed random variable and let $a > 0$. Show that the random variable Y given by

$$Y := \begin{cases} X & \text{if } |X| < a, \\ -X & \text{if } |X| \geq a, \end{cases}$$

has again the $\mathcal{N}(0,1)$ distribution, and find an expression for $\rho(a) = \mathrm{Cov}[X,Y]$ in terms of the density function ϕ of X. Does the pair (X,Y) have a bivariate normal distribution?

2. (Properties of conditional expectations).
Let X, Y be non-negative random variables on $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F} \subseteq \mathcal{A}$ be a σ–algebra. Show that the following assertions are true for any version of the conditional expectations:

a) $E[\,E[X|\mathcal{F}]\,] \;=\; E[X] \qquad P\text{–a.s.}$

b) If $\sigma(X)$ is independent of $\mathcal{F}$, then

$$E[X|\mathcal{F}] \;=\; E[X] \qquad P\text{–a.s.}$$

c) If Y is $\mathcal{F}$–measurable, then

$$E[Y \cdot X|\mathcal{F}] \;=\; Y \cdot E[X|\mathcal{F}] \qquad P\text{–a.s.}$$

d) $E[\lambda X + \mu Y|\mathcal{F}] \;=\; \lambda\, E[X|\mathcal{F}] + \mu\, E[Y|\mathcal{F}] \qquad P\text{–a.s. for all } \lambda, \mu \in \mathbb{R}.$

3. (Martingales).
Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be an increasing sequence of σ–algebras $\mathcal{F}_n \subseteq \mathcal{A}$. Intuitively, $\mathcal{F}_n$ could describe the "information" on a random experiment that is available at time n. A real-valued stochastic process X_n $(n = 0, 1, 2, \ldots)$ on $(\Omega, \mathcal{A}, P)$ is called a *martingale* with respect to $(\mathcal{F}_n)$ if for any $n \geq 0$:

(i) X_n is $\mathcal{F}_n$–measurable and integrable, and

(ii) $E[X_{n+1} - X_n | \mathcal{F}_n] \; = \; 0 \qquad P$–a.s. $\qquad$ *(„fair game")*

Show that:

a) If Y_i $(i \geq 0)$ are independent and integrable with $E[Y_i] = 0$, then

$$S_n \;=\; Y_1 + Y_2 + \cdots + Y_n$$

is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

b) For all random variables $X \in \mathcal{L}^1$, the successive prediction values

$$X_n \;=\; E[X|\mathcal{F}_n]$$

form a martingale.

c) If (X_n) is a martingale with $X_n \in \mathcal{L}^2$, then the increments $\Delta X_n := X_n - X_{n-1}$ are pairwise uncorrelated. State a weak law of large numbers.

4. (Branching with immigration).
Each generation of a branching process (with a single progenitor) is augmented by a random number of immigrants who are indistinguishable from the other members of the population. Suppose that the numbers of immigrants in different generations are independent of each other and of the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function G_n of the size of the nth generation satisfies

$$G_{n+1}(s) \;=\; G_n(G(s))H(s) \;,$$

where G is the probability generating function of a typical family of offspring.

5. (*Additional problem, optional).
Let $X_0, X_1, \ldots : \Omega \to \mathbb{R}$ be i.i.d. random variables with density function f and distribution function F, and let

$$N \;=\; \min\{n \geq 1 : X_n > X_0\}\,.$$

Show that the distribution function of X_N is given by $F + (1 - F)\log(1 - F)$.